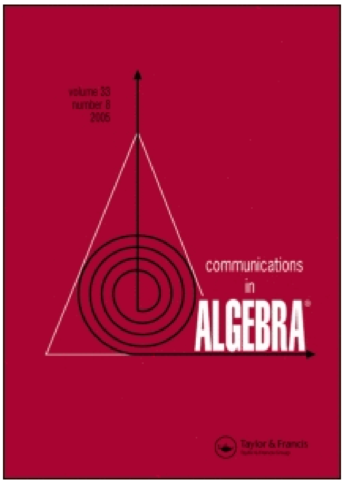

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Nicola Mazzari[a]
[a] Department of Mathematics, Università degli Studi di Padova, Padova, Italy

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EXTENSIONS OF FORMAL HODGE STRUCTURES

Nicola Mazzari
Department of Mathematics, Università degli Studi di Padova, Padova, Italy

We define and study the properties of the category FHS_n of formal Hodge structure of level $\leq n$ following the ideas of Barbieri-Viale who discussed the case of level ≤ 1. As an application, we describe the generalized Albanese variety of Esnault, Srinivas, and Viehweg via the group Ext^1 in FHS_n. This formula generalizes the classical one to the case of proper but not necessarily smooth complex varieties.

Key Words: Albanese; Hodge structures.

2000 Mathematics Subject Classification: 14F (14C30).

INTRODUCTION

The aim of this work is to develop the program proposed by Bloch, Barbieri-Viale, and Srinivas [1, 2] of generalizing Deligne mixed Hodge structures providing a new cohomology theory for complex algebraic varieties. In other words, to construct and study cohomological invariants of (proper) algebraic schemes over $\mathbb{C}$ which are finer than the associated mixed Hodge structures in the case of singular spaces. For any natural number $n > 0$ (the level), we construct an abelian category, FHS_n, and a family of functors

$$\mathrm{H}_\sharp^{n,k} : (\mathsf{Sch}/\mathbb{C})^\circ \to \mathsf{FHS}_n \qquad 1 \leq k \leq n$$

such that:

1. The category MHS_n of mixed Hodge structure of level $\leq n$ is a full subcategory of FHS_n;
2. There is a forgetful functor $f : \mathsf{FHS}_n \to \mathsf{MHS}_n$ s.t. $f(\mathrm{H}_\sharp^{n,k}(X)) = H^n(X)$ (functorially in X) is the usual mixed Hodge structure on the Betti cohomology of X, i.e., $\mathrm{H}^n(X) := \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{Z})$.

Roughly speaking, the sharp cohomology objects $\mathrm{H}_\sharp^{n,k}(X)$ consist of the singular cohomology groups $\mathrm{H}^n(X_{\mathrm{an}}, \mathbb{Z})$, with their mixed Hodge structure, plus some extra structure. We remark that $\mathrm{H}_\sharp^{n,k}(X)$ is completely determined by the mixed

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Address correspondence to Nicola Mazzari, Department of Mathematics, Università degli Studi di Padova, Padova, Italy; E-mail: nclmzzr@gmail.com

Hodge structure on $\mathrm{H}^n(X)$ when X is proper and smooth; the extra structure shows up only when X is not proper or singular.

The motivating example is the following one. Let X be a proper algebraic scheme over $\mathbb{C}$. Denote $\mathrm{H}^i(X) := \mathrm{H}^i(X_{\mathrm{an}}, \mathbb{Z})$, $\mathrm{H}^i(X)_{\mathbb{C}} := \mathrm{H}^i(X)\otimes\mathbb{C}$, and let $\mathrm{H}^{i,j}_{\mathrm{dR}}(X) := \mathrm{H}^i(X_{\mathrm{an}}, \Omega^{<j})$ be the truncated analytic De Rham cohomology of X. Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i(X) \arrow[r] \arrow[dr] & \mathrm{H}^i(X)_{\mathbb{C}}/F^i \arrow[r] & \mathrm{H}^i(X)_{\mathbb{C}}/F^{i-1} \arrow[r] & \cdots \arrow[r] & \mathrm{H}^i(X)_{\mathbb{C}}/F^1 \\
& \mathrm{H}^{i,i}_{\mathrm{dR}}(X) \arrow[r] \arrow[u, "\pi_i"] & \mathrm{H}^{i,i-1}_{\mathrm{dR}}(X) \arrow[r] \arrow[u, "\pi_{i-1}"] & \cdots \arrow[r] & \mathrm{H}^{i,1}_{\mathrm{dR}}(X), \arrow[u, "\pi_1"]
\end{tikzcd}$$

where the $\mathbb{C}$-linear maps π_j are surjective. This diagram is the formal Hodge structure $\mathrm{H}^{i,i}_{\sharp}(X)$ (or simply $\mathrm{H}^i_{\sharp}(X)$).

Note that this definition is compatible with the theory of formal Hodge structures of level ≤ 1 developed by Barbieri-Viale (See [2]). He defined $\mathrm{H}^1_{\sharp}(X)$ as the generalized Hodge realization of $\mathrm{Pic}^0(X)$, i.e., $\mathrm{H}^1_{\sharp}(X) := T_{\oint}(\mathrm{Pic}^0(X))$, which is explicitly represented by the diagram

$$\begin{tikzcd}
\mathrm{H}^1(X) \arrow[r] \arrow[dr] & \mathrm{H}^1(X)_{\mathbb{C}}/F^1 \\
& \mathrm{H}^{1,1}_{\mathrm{dR}}(X) \arrow[u, "\pi_1"]
\end{tikzcd} .$$

As an application of this theory, we can express the Albanese variety of Esnault, Srinivas, and Viehweg [6] using ext-groups. Precisely, let X be a proper, irreducible, algebraic scheme over $\mathbb{C}$. Let $d = \dim X$, and denote by $\mathrm{H}^{2d-1,d}_{\sharp}(X)$ the formal Hodge structure represented by the following diagram:

$$\begin{tikzcd}
\mathrm{H}^{2d-1}(X) \arrow[r] \arrow[dr, "h"] & \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^d \arrow[r] & \cdots \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^1 \\
& \mathrm{H}^{2d-1,d}_{\mathrm{dR}}(X) \arrow[r] \arrow[u] & \cdots \mathrm{H}^{2d-1,1}_{\mathrm{dR}}(X) \arrow[u]
\end{tikzcd} .$$

Then there is an isomorphism of complex Lie groups

$$\mathrm{ESV}(X)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{FHS}_d}\big(\mathbb{Z}(-d), \mathrm{H}^{2d-1,d}_{\sharp}(X)\big),$$

where $\mathrm{ESV}(X)$ is the generalized Albanese of [6]. Note that this formula generalizes the classical one

$$\mathrm{Alb}(X)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{MHS}}\big(\mathbb{Z}(-d), \qquad \mathrm{H}^{2d-1}(X)\big),$$

which follows from the work of Carlson (see [4]).

1. FORMAL HODGE STRUCTURES

We simply call a *formal group* a commutative group of the form $H = H^o \times H_{\text{et}}$, where H_{et} is a finitely generated abelian group and H^o is a finite dimensional $\mathbb{C}$-vector space. We denote by FrmGrp the category with objects formal groups and *morphisms* $f = (f^o, f_{\text{et}}) : H \to H'$, where $f^o : H^o \to H'^o$ is $\mathbb{C}$-linear and $f_{\text{et}} : H_{\text{et}} \to H'_{\text{et}}$ is $\mathbb{Z}$-linear.

We denote the category of mixed Hodge structures of level $\leq l$ (i.e., of type $\{(n, m) \mid 0 \leq n, m \leq l\}$) by $\mathsf{MHS}_l = \mathsf{MHS}_l(0)$, for $l \geq 0$. Also, we define the category $\mathsf{MHS}_l(n)$ to be the full subcategory of MHS whose objects are $H_{\text{et}} \in \mathsf{MHS}$ such that $H_{\text{et}} \otimes \mathbb{Z}(-n)$ is in $\mathsf{MHS}_l(0)$.

Let $\mathsf{Vec} = \mathsf{Vec}_1$ be the category of finite dimensional complex vector spaces and $n > 0$ be an integer. We define the category Vec_n, as follows. The objects are diagrams of $n - 1$ composable arrows of Vec denoted by

$$V : V_n \xrightarrow{v_n} V_{n-1} \xrightarrow{v_{n-1}} V_{n-2} \to \cdots \to V_1.$$

Let $V, V' \in \mathsf{Vec}_n$, a *morphism* $f : V \to V'$ is a family $f_i : V_i \to V'_i$ of $\mathbb{C}$-linear maps such that

$$\begin{tikzcd} V_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & V_i \arrow[d, "f_i"] \\ V'_{i+1} \arrow[r] & V'_i \end{tikzcd}$$

is commutative for all $1 \leq i \leq n$.

Definition 1.1 (Level $= 0$). We define the category of *formal Hodge structures of level* 0 (twisted by k), $\mathsf{FHS}_0(k)$ as follows: the objects are formal groups H such that H_{et} is a pure Hodge structure of type $(-k, -k)$; morphism are maps of formal groups.

Equivalently, $\mathsf{FHS}_0(k)$ is the product category $\mathsf{MHS}_0(k) \times \mathsf{Vec}$.

Definition 1.2 (Level $\leq n$). Fix $n > 0$ an integer. We define a *formal Hodge structure of level* $\leq n$ (or a *n-formal Hodge structure*) to be the data of:

(i) A formal group H (over $\mathbb{C}$) carrying a mixed Hodge structure on the étale component, (H_{et}, F, W), of level $\leq n$. Hence, we get $F^{n+1}H_{\mathbb{C}} = 0$ and $F^0 H_{\mathbb{C}} = H_{\mathbb{C}}$, where $H_{\mathbb{C}} := H_{\text{et}} \otimes \mathbb{C}$.
(ii) A family of fin. gen. $\mathbb{C}$-vector spaces V_i, for $1 \leq i \leq n$.
(iii) A commutative diagram of abelian groups

$$\begin{tikzcd} H_{\text{et}} \arrow[r, "c"] \arrow[dr, "h_{\text{et}}"] & H_{\mathbb{C}}/F^n \arrow[r] & H_{\mathbb{C}}/F^{n-1} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^1 \\ H^o \arrow[r, "h^o"'] & V_n \arrow[u, "\pi_n"] \arrow[r, "v_n"'] & V_{n-1} \arrow[u, "\pi_{n-1}"] \arrow[r, "v_{n-1}"'] & \cdots \arrow[r] & V_1 \arrow[u, "\pi_1"] \end{tikzcd}$$

such that π_i, h^o are $\mathbb{C}$-linear maps.

We denote this object by (H, V) or (H, V, h, π). Note that $V = \{V_n \to \cdots \to V_1\}$ can be viewed as an object of Vec_n.

The map $h = (h_{\text{et}}, h^o) : H \to V_n$ is called *augmentation* of the given formal Hodge structure. A *morphism* of n-formal Hodge structures is a pair (f, ϕ) such that: $f : H \to H'$ is a morphism of formal groups; f induces a morphism of mixed Hodge structures f_{et}; $\phi_i : V_i \to V_i'$ is a family of $\mathbb{C}$-linear maps; $\phi : V \to V'$ is a morphism in Vec_n; and (f, ϕ) are compatible with the above structure, i.e., such that the following diagram commutes:

$$\begin{tikzcd}
 & & & H'_{\text{et}} \arrow[r] \arrow[dr, "h'_{\text{et}}"] & H'_{\mathbb{C}}/F \\
H_{\text{et}} \arrow[r] \arrow[urrr, "f_{\text{et}}"] \arrow[dr, "h_{\text{et}}"'] & H'_{\mathbb{C}}/F \arrow[urrr, "\bar{f}_{\mathbb{C}}"] & & (H')^o \arrow[r, "(h')^o"'] & V' \arrow[u, "\pi'"'] \\
H^o \arrow[r, "h^o"'] \arrow[urrr, "f^o"] & V \arrow[u, "\pi"] \arrow[urrr, "\phi"'] & & &
\end{tikzcd}.$$

We denote this category by $\mathsf{FHS}_n = \mathsf{FHS}_n(0)$.

Remark 1.3. Note that the commutativity of the diagram (iii) of the above definitions implies that the maps π_i are surjective. In fact, after tensor by $\mathbb{C}$ we get that the composition $\pi_n \circ h_{\mathbb{C}}$ is the canonical projection $H_{\mathbb{C}} \to H_{\mathbb{C}}/F^n$: hence π_n is surjective. Similarly, we obtain the surjectivity of π_i for all i.

Example 1.4 (Sharp Cohomology of a Curve)**.** Let $U = X \setminus D$ be a complex projective curve minus a finite number of points. Then we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^1(U) \arrow[r] \arrow[dr] & \mathrm{H}^1(U)_{\mathbb{C}}/F^1 \\
\mathrm{Ker}(\mathrm{H}^{1,1}_{\mathrm{dR}}(X) \to \mathrm{H}^{1,1}_{\mathrm{dR}}(U)) \arrow[r] & \mathrm{H}^{1,1}_{\mathrm{dR}}(X) \arrow[u, "\pi_1"]
\end{tikzcd}$$

representing a formal Hodge structure of level ≤ 1.

Remark 1.5 (Twisted FHS)**.** In a similar way, one can define the category $\mathsf{FHS}_n(k)$, whose object are represented by diagrams

$$\begin{tikzcd}
H_{\text{et}} \arrow[r] \arrow[dr, "h_{\mathbb{Z}}"] & H_{\mathbb{C}}/F^{n-k} \arrow[r] & H_{\mathbb{C}}/F^{n-1-k} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^{1-k} \\
H^o \arrow[r, "h^o"'] & V_{n-k} \arrow[r, "v_{n-k}"'] \arrow[u, "\pi_{n-k}"] & V_{n-k-1} \arrow[r, "v_{n-k-1}"'] \arrow[u, "\pi_{n-k-1}"] & \cdots \arrow[r] & V_{1-k} \arrow[u, "\pi_{1-k}"]
\end{tikzcd}$$

where H_{et} is an object of $\mathsf{MHS}_n(k)$.

Hence the Tate twist $H_{\text{et}} \mapsto H_{\text{et}} \otimes \mathbb{Z}(k)$ induces an equivalence of categories

$$\mathsf{FHS}_n(0) \to \mathsf{FHS}_n(k) \qquad (H, V) \mapsto (H(k), V(k)),$$

where $H(k) = H_{\text{et}} \otimes \mathbb{Z}(k) \times H^o$ and $V(k)$ is obtained by V shifting the degrees, i.e., $V(k)_i = V_{i+k}$, for $1 - k \leq i \leq n - k$.

Example 1.6 (Level ≤ 1)**.** According to the above definition a 1-formal Hodge structure twisted by 1 is represented by a diagram

$$\begin{tikzcd} H_{\text{et}} \arrow[r] \arrow[dr, "h_{\text{et}}"] & H_{\mathbb{C}}/F^0 \\ H^o \arrow[r, "h^o"'] & V_0 \arrow[u, "\pi_0"'] \end{tikzcd},$$

where is (H_{et}, F, W) be a mixed Hodge structure of level ≤ 1 (twisted by $\mathbb{Z}(1)$), i.e., of type $[-1, 0] \times [-1, 0] \subset \mathbb{Z}^2$ (recall that this implies $F^1 H_{\mathbb{C}} = 0$ and $F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}$). If we further assume that H_{et} carries a mixed Hodge structure such that $\mathrm{gr}^W_{-1} H_{\text{et}}$ is polarized, we get the category studied in [2].

Proposition 1.7 (Properties of FHS)**.**

(i) *The category* FHS_n *is an abelian category.*
(ii) *The forgetful functor* $(H, V) \mapsto H$ (*resp.*, $(H, V) \mapsto V$) *is an exact functor with values in the category of formal groups* (*resp., the category* Vec_n).
(iii) *There exists a full and thick embedding* $\mathsf{MHS}_l(0) \to \mathsf{FHS}_l(0)$ *induced by* $(H_{\text{et}}, F, W) \mapsto (H_{\text{et}}, V_i = H_{\mathbb{C}}/F^i)$.
(iv) *There exists a full and thick embedding* $\mathsf{Vec}_l(0) \to \mathsf{FHS}_l(0)$ *induced by* $V \mapsto (0, V)$.

Proof. (i) It follows from the fact that we can compute kernels, co-kernels, and direct sum component-wise.

(ii) It follows by (i).

(iii) Let $(f, \phi) : (H_{\text{et}}, H_{\mathbb{C}}/F) \to (H'_{\text{et}}, H'_{\mathbb{C}}/F)$ be a morphism in FHS_n. Then by definition, for any $1 \leq i \leq n$, there is a commutative diagram

$$\begin{tikzcd} H_{\mathbb{C}}/F^i \arrow[r, "\phi_i"] \arrow[d, "\mathrm{id}"'] & H'_{\mathbb{C}}/F^i \arrow[d, "\mathrm{id}"] \\ H_{\mathbb{C}}/F^i \arrow[r, "\bar{f}_i"'] & H'_{\mathbb{C}}/F^i \end{tikzcd},$$

where $\bar{f}_i(x + F^i H_{\mathbb{C}}) = f(x) + F^i H'_{\mathbb{C}}$ is the map induce by f: it is well defined because the morphisms of mixed Hodge structures are strictly compatible w.r.t. the Hodge filtration. Hence ϕ is completely determined by f.

(iv) It is a direct consequence of the definition of FHS_n. $\square$

Lemma 1.8. *Fix $n \in \mathbb{Z}$. The following functor*

$$\mathsf{MHS} \to \mathsf{Vec}, \qquad (H_{\mathrm{et}}, W, F) \mapsto H_{\mathbb{C}}/F^n$$

is an exact functor.

Proof. This follows from [5, §1.2.10]. □

1.1. Subcategories of FHS_n

Let (H, V) be a formal Hodge structure of level $\leq n$. It can be visualized as a diagram

$$\begin{tikzcd}
H_{\mathrm{et}} \arrow[r] \arrow[dr, "h_{\mathrm{et}}"] & H_{\mathbb{C}}/F^n \arrow[r] & H_{\mathbb{C}}/F^{n-1} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^1 \\
H^o \arrow[r, "h^o"'] & V_n \arrow[u, "\pi_n"] \arrow[r, "v_n"'] & V_{n-1} \arrow[u, "\pi_{n-1}"] \arrow[r, "v_{n-1}"'] & \cdots \arrow[r] & V_1 \arrow[u, "\pi_1"] \\
 & V_n^o \arrow[u] \arrow[r] & V_{n-1}^o \arrow[u] \arrow[r] & \cdots \arrow[r] & V_1^o \arrow[u]
\end{tikzcd} ,$$

where $V_i^o := \mathrm{Ker}(\pi_i : V_i \to H_{\mathbb{C}}/F^i)$. We can consider the following n-formal Hodge structures:

1. $(H, V)_{\mathrm{et}} := (H_{\mathrm{et}}, V/V^o)$, called the *étale part* of (H, V);
2. $(H, V)_\times := (H, V/V^o)$, where the augmentation $H \to H_{\mathbb{C}}/F^n = V_n/V_n^o$ is the composite $\pi_n \circ h$.

We say that (H, V) is *étale* (resp., *connected*) if $(H, V) = (H, V)_{\mathrm{et}}$ (resp., $(H, V)_{\mathrm{et}} = 0$). Also we say that (H, V) is *special* if $h^o : H^o \to V_n$ factors through V_n^o. We will denote by $\mathsf{FHS}_{n,\mathrm{et}}$ (resp., FHS_n^o, FHS_n^s) the full subcategory of FHS_n whose objects are étale (resp., connected, special). Note that by construction the category of étale formal Hodge structure $\mathsf{FHS}_{n,\mathrm{et}}$ is equivalent to MHS_n, by abuse of notation, we will identify these two categories.

Proposition 1.9 (Canonical Decomposition)**.**

(i) *Let $(H, V) \in \mathsf{FHS}_n$ $(n > 0)$, then there are two canonical exact sequences*

$$0 \to (0, V^o) \to (H, V) \to (H, V)_\times \to 0; \qquad 0 \to (H, V)_{\mathrm{et}} \to (H, V)_\times \to (H^o, 0) \to 0.$$

(ii) *The augmentation $h^o : H^o \to V_n$ factors trough V_n^o $\iff$ there is a canonical exact sequence*

$$0 \to (H, V)^o \to (H, V) \to (H, V)_{\mathrm{et}} \to 0,$$

where $(H, V)^o := (H^o, V^o)$.

Proof. (i) Let $(0, \theta) : (0, V^o) \to (H, V)$ be the canonical inclusion. By 1.7 $\mathrm{Coker}(0, \theta)$ can be calculated in the product category $\mathsf{FrmGrp} \times \mathsf{Vec}_n$, i.e., $\mathrm{Coker}(0, \theta) = \mathrm{Coker}\, 0 \times \mathrm{Coker}\, \theta = H \times V/V^o$ and the augmentation $H \to H_{\mathbb{C}}/F^n$ is the composition $H \xrightarrow{h} V_n \xrightarrow{\pi_n} H_{\mathbb{C}}/F^n$.

For the second exact sequence, consider the natural projection $p^o : H \to H^o$. It induces a morphism $(p^o, 0) : (H, V)_\times \to (H^o, 0)$. Using the same argument as above, we get $\mathrm{Ker}(p^o, 0) = \mathrm{Ker}\, p^o \times \mathrm{Ker}\, 0 = H_{\mathrm{et}} \times V/V^0$ as an object of $\mathsf{FrmGrp} \times \mathsf{Vec}_n$. From this follows the second exact sequence.

(ii) By the definition of a morphism of formal Hodge structures (of level $\leq n$), we get that the canonical map, in the category $\mathsf{FrmGrp} \times \mathsf{Vec}_n$, $(p_{\mathbb{Z}}, \pi) : H \times V \to H_{\mathrm{et}} \times V/V^o$ induces a morphism of formal Hodge structures $\iff$ the following diagram commutes:

$$
\begin{tikzcd}
H \arrow[r, "p_{\mathbb{Z}}"] \arrow[d, "h"'] & H_{\mathbb{Z}} \arrow[d] \\
V_n \arrow[r, "\pi_n"'] & H_{\mathbb{C}}/F^n
\end{tikzcd}
$$

i.e., $\pi_n h(x, y) = y \mod F^n H_{\mathbb{C}}$ for all $x \in H^o$, $y \in H_{\mathrm{et}} \iff h^o(x) = 0$. □

Remark 1.10. With the above notations, consider the map $(p^o, 0) : H \times V \to H^o \times 0$. Note that this is a morphism of formal Hodge structure $\iff V^0 = 0 \iff (H, V) = (H, V)_\times$.

Remark 1.11. For $n = 0$, we can also use the same definitions, but the situation is much easier. In fact a formal structure of level 0 is just a formal group H; hence, there is a split exact sequence

$$0 \to H^o \to H \to H_{\mathrm{et}} \to 0$$

in $\mathsf{FHS}_0(0)$.

Corollary 1.12. *Let $\mathfrak{K}_0(\mathsf{FHS}_n)$ be the Grothendieck group (see [10, Def. A.4]) associated to the abelian category FHS_n. Then*

$$
\begin{aligned}
\mathfrak{K}_0(\mathsf{FHS}_n) &= \mathfrak{K}_0(\mathsf{Vec}) \times \mathfrak{K}_0(\mathsf{Vec}_n) \times \mathfrak{K}_0(\mathsf{MHS}_n) \\
&\cong \{(f, g) \in \mathbb{Z}[t] \times \mathbb{Z}[u, v] \mid \deg_t f, \deg_u g, \deg_v g \leq n, g(u, v) = g(v, u)\}.
\end{aligned}
$$

Proof. It follows easily by (i) of 1.9. □

By 1.7 there exists a canonical embedding $\mathsf{MHS}_n \subset \mathsf{FHS}_n$ (resp., $\mathsf{Vec}_n \subset \mathsf{FHS}_n$). It is easy to check that this embedding gives, in the usual way, a full embedding when passing to the associated homotopy categories, i.e.,

$$K(\mathsf{MHS}_n) \subset K(\mathsf{FHS}_n), \qquad \text{resp. } K(\mathsf{Vec}_n) \subset K(\mathsf{FHS}_n). \tag{1}$$

With the following lemma, we can prove that we have an embedding when passing to the associated derived categories.

Lemma 1.13. *Let $\mathsf{A}' \subset A$ be a full embedding of categories. Let S be a multiplicative system in A and S' be its restriction to A'. Assume that one of the following conditions*:

(i) *For any $s : A' \to A$ (where $A' \in \mathsf{A}'$, $A \in \mathsf{A}$, $s \in S$), there exists a morphism $f : A \to B'$ such that $B' \in \mathsf{A}'$ and $f \circ s \in S$;*
(ii) *The same as (i) with the arrow reversed.*

Then the localization $\mathsf{A}'_{S'}$ is a full subcategory of A_S.

Proof. [7, 1.6.5]. □

Proposition 1.14. *There is a full embedding of categories $D(\mathsf{MHS}_n) \subset D(\mathsf{FHS}_n)$ (resp., $D(\mathsf{Vec}_n) \subset D(\mathsf{FHS}_n)$).*

Proof. We will prove only the case involving MHS_n, the other one is similar. First note that, similarly to (1), there is a full embedding $K(\mathsf{FHS}_{n,\times}) \subset K(\mathsf{FHS}_n)$, where $\mathsf{FHS}_{n,\times}$ is the full subcategory of FHS_n with objects (H, V) such that $(H, V) = (H, V)_\times$ (see 1.9). Now using (i) of Lemma 1.13 and the first exact sequence of 1.9, we get a full embedding $D(\mathsf{FHS}_{n,\times}) \subset D(\mathsf{FHS}_n)$.

Then consider the canonical embedding $\mathsf{MHS}_n \subset \mathsf{FHS}_{n,\times}$. Again we get a full embedding of triangulated categories $K(\mathsf{MHS}_n) \subset K(\mathsf{FHS}_{n,\times})$. Now using (ii) of Lemma 1.13 and the second exact sequence of 1.9, we get a full embedding $D(\mathsf{FHS}_{n,\times}) \subset D(\mathsf{FHS}_n)$. □

1.2. Adjunctions

Proposition 1.15. *The following adjunction formulas hold*:

(i) $\mathrm{Hom}_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}}) \cong \mathrm{Hom}_{\mathsf{FHS}_n}((H, V), (H'_{\mathrm{et}}, H'_{\mathbb{C}}/F))$ *for all* $(H, V) \in \mathsf{FHS}^s_n$ (*i.e., special*), $H'_{\mathrm{et}} \in \mathsf{MHS}_n$;
(ii) $\mathrm{Hom}_{\mathsf{FHS}_n}((H^o, V), (H', V')) \cong \mathrm{Hom}_{\mathsf{FHS}_n}((H^o, V), ((H')^o, (V')^o))$ *for all* $(H^o, V) \in \mathsf{FHS}^o_n$ (*i.e., connected*), $(H', V') \in \mathsf{FHS}^s_n$.

Proof. The proof is straightforward. Explicitly:

(i) Let $(H, V) \in \mathsf{FHS}^s_n$, $H'_{\mathrm{et}} \in \mathsf{MHS}_n$. By definition a morphism $(f, \phi) \in \mathrm{Hom}_{\mathsf{FHS}_n}((H, V), (H'_{\mathrm{et}}, H'_{\mathbb{C}}/F))$ is a morphism of formal groups $f : H \to H'$ such that f_{et} is a morphism of mixed Hodge structures, hence $f = f_{\mathrm{et}}$, and $\phi : V \to H'_{\mathbb{C}}/F$ is subject to the condition $f/F \circ \pi = \phi$. Then the association $(f, \phi) \mapsto f_{\mathrm{et}} \in \mathrm{Hom}_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}})$ is an isomorphism.

(ii) Let $(H^o, V) \in \mathsf{FHS}^o_n$, $(H', V') \in \mathsf{FHS}^s_n$.

A morphism (f, ϕ) in $\mathrm{Hom}_{\mathsf{FHS}_n}((H^o, V), (H', V'))$ is of the form $f = f^o : H^o \to (H')^o$, $\phi : V \to V'$ must factor through $(V')^o$ because $\pi' \circ \phi = \pi \circ f/F = 0$. □

1.3. Different Levels

Any mixed Hodge structure of level $\leq n$ (say in $\mathsf{MHS}_n(0)$) can also be viewed as an object of $\mathsf{MHS}_m(0)$ for any $m > n$. This give a sequence of full embeddings

$$\mathsf{MHS}_0 \subset \mathsf{MHS}_1 \subset \cdots \subset \mathsf{MHS}.$$

In this section, we want to investigate the analogous situation in the case of formal Hodge structures.

Consider the two functors $\imath, \eta : \mathsf{Vec}_n \to \mathsf{Vec}_{n+1}$ defined as follows:

$$\imath(V) : \imath(V)_{n+1} = V_n \overset{\mathrm{id}}{\to} \imath(V)_n = V_n \overset{v_n}{\to} \cdots \to V_1,$$
$$\eta(V) : \eta(V)_{n+1} = 0 \overset{0}{\to} \imath(V)_n = V_n \overset{v_n}{\to} \cdots \to V_1.$$

Proposition 1.16. *The functors $\imath, \eta$ are full and faithful. Moreover, the essential image of $\imath$ (resp., η) is a thick subcategory.*[1]

Proof. To check that $\imath, \eta$ are embeddings, it is straightforward. We prove that the essential image of $\imath$ (resp., η) is closed under extensions only in case $n = 2$ just to simplify the notations.

First consider an extension of ηV by $\eta V'$ in Vec_3

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & V_3'' \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_2' \arrow[r] \arrow[d] & V_2'' \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_1' \arrow[r] & V_1'' \arrow[r] & V_1 \arrow[r] & 0 .
\end{tikzcd}$$

Then it follows that $V_3'' = 0$.

Now consider an extension of $\imath V$ by $\imath V'$ in Vec_3

$$\begin{tikzcd}
0 \arrow[r] & V_2' \arrow[r] \arrow[d, "\mathrm{id}"] & V_3'' \arrow[r] \arrow[d, "v"] & V_2 \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & V_2' \arrow[r] \arrow[d] & V_2'' \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_1' \arrow[r] & V_1'' \arrow[r] & V_1 \arrow[r] & 0 .
\end{tikzcd}$$

Then v is an isomorphism (by the snake lemma). It follows that V'' is isomorphic, in Vec_3, to an object of $\imath\mathsf{Vec}_2$. To check that the essential image of $\imath$ (resp., η) is closed under kernels and cokernels is straightforward. $\square$

[1]By thick we mean a subcategory closed under kernels, co-kernels, and extensions.

Remark 1.17. The category of complexes of objects of Vec concentrated in degrees $1, \ldots, n$ is a full subcategory of Vec_n. Moreover, the embedding induces an equivalence of categories for $n = 1$ and 2, but for $n > 2$ the embedding is not thick.

Example 1.18 ($\mathsf{FHS}_1 \subset \mathsf{FHS}_2$). The basic construction is the following one: Let (H, V) be a 1-fhs; we can associate a 2-fhs (H', V') represented by a diagram of the following type:

$$\begin{tikzcd}
H'_{\mathrm{et}} \arrow[r] \arrow[dr, "h'_{\mathbb{Z}}"'] & H'_{\mathbb{C}}/F^2 \arrow[r] & H'_{\mathbb{C}}/F^1 \\
(H')^o \arrow[r, "(h')^o"'] & V'_2 \arrow[u, "\pi'_2"'] \arrow[r, "v'_2"'] & V'_1 \arrow[u, "\pi'_1"']
\end{tikzcd}.$$

Take $H'_{\mathrm{et}} := H_{\mathrm{et}}$, then $H'_{\mathbb{C}}/F^2 = H_{\mathbb{C}}$, $H'_{\mathbb{C}}/F^1 = H_{\mathbb{C}}/F^1$, and the augmentation h'_{et} is the canonical inclusion; let $V'_1 := V_1$, $\pi'_1 := \pi_1$, and define V'_2, π'_2, v'_2 via fiber product

$$\begin{tikzcd}
V'_2 \arrow[r, "\pi'_2"] \arrow[d, "v'_2"'] & H_{\mathbb{C}} \arrow[d] \\
V_1 \arrow[r, "\pi_1"'] & H_{\mathbb{C}}/F^1.
\end{tikzcd}$$

Hence V'_2 fits in the following exact sequences:

$$0 \to F^1 H_{\mathbb{C}} \to V'_2 \to V_1 \to 0; \qquad 0 \to V_1^0 \to V'_2 \to H_{\mathbb{C}} \to 0.$$

Finally, we define $(h')^o : (H')^o \to V'_2$ again via fiber product

$$\begin{tikzcd}
(H')^o \arrow[r, "(h')^o"] \arrow[d] & V'_2 \arrow[d, "v'_2"] \\
H^o \arrow[r, "h^o"'] & V_1
\end{tikzcd}.$$

Hence, we get the exact sequence

$$0 \to F^1 H_{\mathbb{C}} \to (H')^o \to H^o \to 0.$$

By induction, it is easy to extend this construction. We have the following result.

Proposition 1.19. *Let $n, k > 0$. Then there exists a faithful functor*

$$\iota = \iota_k : \mathsf{FHS}_n \to \mathsf{FHS}_{n+k}.$$

Moreover, $\imath$ induces an equivalence between FHS_n and the subcategory of FHS_{n+k} whose objects are (H, V) such that:

(a) *H_{et} is of level $\leq n$. Hence $F^{n+1}H_{\mathbb{C}} = 0$ and $F^0 H_{\mathbb{C}} = H_{\mathbb{C}}$;*
(b) *$V_{n+i} = V_{n+1}$ for $1 \leq i \leq k$;*
(c) *There exists a commutative diagram with exact rows*

$$\begin{tikzcd}
F^n \arrow[r] & H_{\mathbb{C}} \arrow[r] & H_{\mathbb{C}}/F^n \\
F^n \arrow[u, "\mathrm{id}"] \arrow[r] \arrow[dr, "\alpha"'] & V_{n+1} \arrow[u, "\pi_{n+1}"'] \arrow[r, "v_{n+1}"] & V_n \arrow[u, "\pi_n"'] , \\
 & H^o \arrow[u, "h^o"'] &
\end{tikzcd}$$

where α is a $\mathbb{C}$-linear map.

Morphisms are those in FHS_{n+k} compatible w.r.t. the diagram in (c).

Proof. The construction of $\imath_k$ is a generalization of that in 1.18. We have $\imath_k = \imath_1 \circ \imath_{k-1}$; hence, it is enough to define $\imath_1$, which is the same as in 1.18 up to a change of subscripts: $n = 1$, $n + 1 = 2$.

To prove the equivalence, we define a quasi-inverse: Let $(H', V') \in \mathsf{FHS}_{n+1}$ and satisfying a, b, c, and $\alpha : F^n H'_{\mathbb{C}} \to (H')^o$ as in the proposition.

Define $(H, V) \in \mathsf{FHS}_n$ in the following way: $H = H'/\alpha(F^n H'_{\mathbb{C}})$; $V_i = V'_i$ for all $1 \leq i \leq n$; $h : H'/\alpha(F^n H'_{\mathbb{C}}) \xrightarrow{\bar{h}'} V'_{n+1} \xrightarrow{v'_{n+1}} V'_n = V_n$, where $\bar{h}' = (h'_{\text{et}}, (h')^o \bmod F^n)$. □

Proposition 1.20. *Let $n, k > 0$ and denote by $\imath_k \mathsf{FHS}_n \subset \mathsf{FHS}_{n+k}$ the essential image of FHS_n (see the previous proposition). Then $\imath_k \mathsf{FHS}_n \subset \mathsf{FHS}_{n+k}$ is an abelian (not full) subcategory closed under kernels, cokernels, and extensions.*

Proof. Straightforward. □

Remark 1.21. Note that $\imath_k \mathsf{FHS}_n \subset \mathsf{FHS}_{n+k}$ it is not closed under subobjects.

Remark 1.22. Let FHS_n^{prp} be the full subcategory of FHS_n whose objects are formal Hodge structures (H, V) with $H^o = 0$.[2] Then $\imath_k$ induces a full and faithful functor

$$\imath = \imath_k : \mathsf{FHS}_n^{prp} \to \mathsf{FHS}_{n+k}^{prp}.$$

Moreover, $\imath_k \mathsf{FHS}_n^{prp} \subset \mathsf{FHS}_{n+k}^{prp}$ is an abelian thick subcategory.

Example 1.23 (Special Structures). For special structures, it is natural to consider the following construction, similar to $\imath_k$ (compare with 1.18). Let (H, V) be a formal Hodge structures of level ≤ 1. Define $\tau(H, V) = (H, V')$ to be the formal Hodge

[2]The superscript *prp* stands for "proper". In fact, the sharp cohomology objects (3.1) of a proper variety have this property.

structure of level ≤ 2 represented by the following diagram

$$\begin{tikzcd}
H_{\mathrm{et}} \arrow[r] \arrow[dr] & H_{\mathbb{C}} \arrow[r, "h_{\mathbb{C}}"] & H_{\mathbb{C}}/F^1 \\
H^o \arrow[r, "(h')^o"'] & V_2' \arrow[u, "\pi_2'"'] \arrow[r, "v_2'"'] & V_1 \arrow[u, "\pi_1"'] ,
\end{tikzcd}$$

where V_2', v_2', $(h')^o$ are defined via fiber product as follows:

$$\begin{tikzcd}
H^o \arrow[dr, dotted, "(h')^o"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "h^o"'] & & \\
& V_2' \arrow[r, "\pi_2'"] \arrow[d, "v_2'"'] & H_{\mathbb{C}} \arrow[d] \\
& V_1 \arrow[r, "\pi_1"'] & H_{\mathbb{C}}/F^1.
\end{tikzcd}$$

Note that the commutativity of the external square is equivalent to say that (H, V) is special. Hence, this construction cannot be used for general formal Hodge structures.

Proposition 1.24. *Let $n, k > 0$ be integers. Then there exists a full and faithful functor*

$$\tau = \tau_k : \mathsf{FHS}_n^s \to \mathsf{FHS}_{n+k}^s.$$

Moreover, the essential image of τ_k, $\tau_k \mathsf{FHS}_n^{spc}$, is the full and thick abelian subcategory of FHS_{n+k}^{spc} with objects (H, V) such that:

(a) *H_{et} is of level $\leq n$. Hence $F^{n+1}H_{\mathbb{C}} = 0$ and $F^0 H_{\mathbb{C}} = H_{\mathbb{C}}$;*
(b) *$V_{n+i} = V_{n+1}$ for $1 \leq i \leq k$;*
(c) *$V_{n+1} = H_{\mathbb{C}} \times_{H_{\mathbb{C}}/F^n} V_n$.*

Proof. Note that $\tau_k = \tau_1 \circ \tau_{k-1}$; hence, it is enough to construct τ_1. Let (H, V) be a special formal Hodge structure of level $\leq n$. Then $\tau_1(H, V)$ is defined as in 1.23 up to change the subscripts $n = 1$, $n + 1 = 2$.

To prove the equivalence, it is enough to construct a quasi-inverse of τ_1. Let (H', V') be a special formal Hodge structure of level $\leq n$ satisfying the conditions a, b, c of the proposition; then, define $(H, V) \in \mathsf{FHS}_n$ as follows: $H := H'$; $V_i := V_i'$ for all $1 \leq i \leq n$; $h = v_{n+1}' \circ h'$.

Thickness follows directly from the exactness of the functors

$$(H, V) \mapsto H_{\mathrm{et}}, \qquad (H, V) \mapsto V^o. \qquad \square$$

Remark 1.25. The functors τ_k, ι_k agree on the full subcategory of FHS_n formed by (H, V) with $H^o = 0$.

2. EXTENSIONS IN FHS_n

2.1. Basic Facts

Example 2.1. We describe the ext-groups for Vec_2. We have the following isomorphism:

$$\phi : \mathrm{Ext}^1_{\mathsf{Vec}_2}(V, V') \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Vec}}(\mathrm{Ker}\, v, \mathrm{Coker}\, v').$$

Explicitly, ϕ associates to any extension class the Ker–Coker boundary map of the snake lemma. To prove it is an isomorphism, we argue as follows. The abelian category Vec_2 is equivalent to the full subcategory C' of $C^b(\mathsf{Vec})$ of complexes concentrated in degree 0, 1. Hence, the group of classes of extensions is isomorphic. Now let $a : A^0 \to A^1$, $b : B^0 \to B^1$ be two complexes of objects of Vec. Then we have

$$\mathrm{Ext}^1_{C'}(A^\bullet, B^\bullet) = \mathrm{Ext}^1_{C^b(\mathsf{Vec})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{D^b(\mathsf{Vec})}(A^\bullet, B^\bullet[1])$$

because C' is a thick subcategory of $C^b(\mathsf{Vec})$.

The category Vec is of cohomological dimension 0. Then $a : A^0 \to A^1$ is quasi-isomorphic to $\mathrm{Ker}\, a \xrightarrow{0} \mathrm{Coker}\, a$, and similarly for $B^\bullet$. It follows that

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathsf{Vec})}(A^\bullet, B^\bullet[1]) &= \mathrm{Hom}_{D^b(\mathsf{Vec})}(\mathrm{Ker}\, a[0] \oplus \mathrm{Coker}\, a[-1], \mathrm{Ker}\, b[1] \oplus \mathrm{Coker}\, b[0]) \\ &= \mathrm{Hom}_{\mathsf{Vec}}(\mathrm{Ker}\, a, \mathrm{Coker}\, b).\end{aligned}$$

As a corollary, we obtain that $\mathrm{Ext}^1_{\mathsf{Vec}_2}(V, -)$ is a right exact functor, and this is a sufficient condition for the vanishing of $\mathrm{Ext}^i_{\mathsf{Vec}_2}(, -)$ for $i \leq 2$ (i.e., Vec_2 is a category of cohomological dimension 1).

Example 2.2. The category Vec_3 is of cohomological dimension 1. We argue as in [9]. Let V be an object of Vec_3. We define the following increasing filtration

$$W_{-2} = \{0 \to 0 \to V_1\}; \qquad W_{-1} = \{0 \to V_2 \to V_1\}; \qquad W_0 = V.$$

Note that morphisms in Vec_3 are compatible w.r.t. this filtration. To prove that $\mathrm{Ext}^2_{\mathsf{Vec}_3}(V, V') = 0$, it is sufficient to show that $\mathrm{Ext}^2_{\mathsf{Vec}_3}(\mathrm{gr}^W_i V, \mathrm{gr}^W_j V') = 0$ for $i, j = -2, -1, 0$ (just use the short exact sequences induced by W, cf. [9, Proof of 2.5]). We prove the case $i = 0$, $j = -2$, leaving to the reader the other cases (which are easier, cf. [9, 2.2–2.4]).

Let $\gamma \in \mathrm{Ext}^2_{\mathsf{Vec}_3}(\mathrm{gr}^W_0 V, \mathrm{gr}^W_{-2} V') = 0$. We can represent γ by an exact sequence in Vec_3 of the type

$$0 \to \mathrm{gr}^W_{-2} V' \to A \to B \to \mathrm{gr}^W_0 V \to 0.$$

Let $C = \mathrm{Coker}(\mathrm{gr}^W_{-2} V' \to A) = \mathrm{Ker}(B \to \mathrm{gr}^W_0 V)$. Then $\gamma = \gamma_1 \cdot \gamma_2$, where $\gamma_1 \in \mathrm{Ext}^1_{\mathsf{Vec}_3}(C, \mathrm{gr}^W_{-2} V')$, $\gamma_2 \in \mathrm{Ext}^1_{\mathsf{Vec}_3}(\mathrm{gr}^W_0 V, C)$. Arguing as in [9, 2.4], we can suppose that $C = \mathrm{gr}^W_{-1} C$; hence,

$$\gamma_1 = [0 \to \mathrm{gr}^W_{-2} V' \to A \to \mathrm{gr}^W_{-1} C \to 0], \qquad \gamma_2 = [0 \to \mathrm{gr}^W_{-1} C \to B \to \mathrm{gr}^W_0 V \to 0].$$

It follows that $A = \{0 \to C_2 \xrightarrow{f_1} V_1'\}$, $B = \{V_3 \xrightarrow{f_2} C_2 \to 0\}$, for some f_1, f_2. Now consider $D = \{V_3 \xrightarrow{f_2} C_2 \xrightarrow{f_1} V_1'\} \in \mathsf{Vec}_3$. Then it is easy to check that

$$\gamma_1 = [0 \to W_{-2}D \to W_{-1}D \to \mathrm{gr}^W_{-1}D \to 0],$$
$$\gamma_2 = [0 \to \mathrm{gr}_{-1}D \to W_0D/W_{-2}D \to \mathrm{gr}^W_0 D \to 0].$$

By [9, Lemma 2.1], $\gamma = 0$.

Proposition 2.3. *Let H_{et} be a mixed Hodge structure of level $\leq n$. We consider it as an étale formal Hodge structure. Let (H', V') be be a formal Hodge structure of level $\leq n$ (for $n > 0$). Then*:

(i) *There is a canonical isomorphism of abelian groups*

$$Ext^1_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}}) \cong Ext^1_{\mathsf{FHS}_n}(H_{\mathrm{et}}, (H', V'/V'^o)).$$

(ii) *For any $i \geq 2$, there is a canonical isomorphism*

$$Ext^i_{\mathsf{FHS}_n}(H_{\mathrm{et}}, (H', V'/V'^o)) \cong Ext^i_{\mathsf{FHS}_n}(H_{\mathrm{et}}, (H'^o, 0)).$$

Proof. This follows easily by the computation of the long exact sequence obtained applying $\mathrm{Hom}_{\mathsf{FHS_n}}(H_{\mathbb{Z}}, -)$ to the short exact sequence

$$0 \to (H', V')_{\mathrm{et}} \to (H', V')_{\times} \to (H'^o, 0) \to 0. \qquad \square$$

Proposition 2.4. *The forgetful functor $(H, V) \mapsto H_{\mathrm{et}}$ induces a surjective morphism of abelian groups*

$$\gamma : Ext^1_{\mathsf{FHS}_n}((H, V), (H', V')) \to Ext^1_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}})$$

for any $(H, V), (H', V')$ with $H_{\mathrm{et}}, H'_{\mathrm{et}}$ free.

Proof. Recall the extension formula for mixed Hodge structures is (see [10, I §3.5])

$$\mathrm{Ext}^1_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}}) \cong \frac{W_0\mathscr{H}om(H_{\mathrm{et}}, H'_{\mathrm{et}})_{\mathbb{C}}}{F^0 \cap W_0(\mathscr{H}om(H_{\mathrm{et}}, H'_{\mathrm{et}})_{\mathbb{C}}) + W_0\mathscr{H}om(H_{\mathrm{et}}, H'_{\mathrm{et}})_{\mathbb{Z}}}. \tag{2}$$

More precisely, we get that any extension class can be represented by $\widetilde{H}_{\mathrm{et}} = (H'_{\mathrm{et}} \oplus H_{\mathrm{et}}, W, F_\theta)$, where the weight filtration is the direct sum $W_iH'_{\mathrm{et}} \oplus W_iH_{\mathrm{et}}$ and $F^i_\theta := F^iH'_{\mathrm{et}} + \theta(F^iH_{\mathrm{et}}) \oplus F^iH_{\mathrm{et}}$, for some $\theta \in W_0\mathscr{H}om(H_{\mathrm{et}}, H'_{\mathrm{et}})_{\mathbb{C}}$. It follows that $\widetilde{H}_{\mathbb{C}}/F^i_\theta = H'_{\mathbb{C}}/F^i \oplus H_{\mathbb{C}}/F^i$. Then we can consider the formal Hodge structure of level $\leq n$ $(\widetilde{H}, \widetilde{V})$ defined as follows: $\widetilde{H}_{\mathrm{et}} = (H'_{\mathrm{et}} \oplus H_{\mathrm{et}}, W, F_\theta)$ as above; $\widetilde{H}^o := (H')^o \oplus H^o$; $\widetilde{V}_i := V'_i \oplus V_i$, $\tilde{v}_i := (v'_i, v_i)$; and $\tilde{h} = (h', h)$. Then it is easy to check that $(\widetilde{H}, \widetilde{V}) \in \mathrm{Ext}^1_{\mathsf{FHS}_n}((H', V'), (H, V))$ and $\gamma(\widetilde{H}, \widetilde{V}) = (H'_{\mathrm{et}} \oplus H_{\mathrm{et}}, W, F_\theta)$. $\square$

Example 2.5 (Infinitesimal Deformation). Let $f : \widehat{X} \to \mathrm{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$ a smooth and projective morphism. Write $X/\mathbb{C}$ for the smooth and projective variety corresponding to the special fiber, i.e., the fiber product

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widehat{X} \arrow[d, "f"] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r] & \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2). \end{tikzcd}$$

Then (see [1, 2.4]) for any i, n there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^{n-i+1}(X_{\mathrm{an}}, \Omega^{i-1}) \arrow[r] \arrow[d, "0"] & \mathrm{H}^n(\widehat{X}_{\mathrm{an}}, \Omega^{<i}) \arrow[r] \arrow[d] & \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{C})/F^i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{H}^{n-i+2}(X_{\mathrm{an}}, \Omega^{i-2}) \arrow[r] & \mathrm{H}^n(\widehat{X}_{\mathrm{an}}, \Omega^{<i-1}) \arrow[r] & \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{C})/F^{i-1} \arrow[r] & 0\,. \end{tikzcd}$$

Hence there is an extension of formal Hodge structures of level $\leq n$

$$0 \to (0, V) \to (\mathrm{H}^n(X), \mathrm{H}^{n,*}_{\mathrm{dR}}(\widehat{X})) \to \mathrm{H}^n(X) \to 0$$

with $V_i = \mathrm{H}^{n-i+1}(X_{\mathrm{an}}, \Omega^{i-1})$ and $v_i = 0$.

Remark 2.6. It is well known that the groups $\mathrm{Ext}^i(A, B)$ vanish in category of mixed Hodge structures for any $i > 1$. It is natural to ask if the groups $\mathrm{Ext}^i_{\mathsf{FHS}_n}((H, V), (H', V'))$ vanish for $i > n$ (up to torsion). In particular, Bloch and Srinivas raised a similar question for special formal Hodge structures (cf. [1]).

The author answered positively this question for $n = 1$ in [9].

2.2. Formal Carlson Theory

Proposition 2.7. *Let A, B torsion-free mixed Hodge structures. Suppose B pure of weight $2p$ and A of weights $\leq 2p - 1$. There is a commutative diagram of complex Lie group*

$$\begin{tikzcd} \mathrm{Ext}^1_{\mathsf{MHS}}(B, A) \arrow[r, "\gamma"] \arrow[dr, "i^*"'] & \mathrm{Hom}_{\mathbb{Z}}(B^{p,p}_{\mathbb{Z}}, J^p(A)) \\ & \mathrm{Ext}^1_{\mathsf{MHS}}(B^{p,p}_{\mathbb{Z}}, A), \arrow[u, "\bar{\gamma}"'] \end{tikzcd}$$

where $\bar{\gamma}$ is an isomorphism; i^* is the surjection induced by the inclusion $i : B^{p,p}_{\mathbb{Z}} \to B$.

Proof. This follows easily from the explicit formula (2). The construction of $\gamma, \bar{\gamma}$ is given in the following remark. Then choosing a basis of $B^{p,p}_{\mathbb{Z}}$, it is easy to check that $\bar{\gamma}$ is an isomorphism. □

Remark 2.8.

(i) Let $\{b_1, \dots, b_n\}$ a $\mathbb{Z}$-basis of $B_{\mathbb{Z}}^{p,p}$. Then $\mathrm{Hom}_{\mathbb{Z}}(B_{\mathbb{Z}}^{p,p}, J^p(A)) \cong \bigoplus_{i=1}^n J^p(A)$, which is a complex Lie group.

(ii) Explicitly, γ can be constructed as follows. Let $x \in \mathrm{Ext}^1_{\mathsf{MHS}}(B, A)$ represented by the extension

$$0 \to A \to H \to B \to 0.$$

Then, apply $\mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), -)$ to the above exact sequence, and consider the boundary of the associated long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), B) \xrightarrow{\partial_x} \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), A) \to \cdots.$$

Note that ∂_x does not depend on the choice of the representative of x; $\mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), B) = B_{\mathbb{Z}}^{p,p}$; $J^p(A) = \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), A)$.
Hence we can define $\gamma(x) := \partial_x \in \mathrm{Hom}_{\mathbb{Z}}(B_{\mathbb{Z}}^{p,p}, J^p(A))$.

(iii) If the complex Lie group $J^p(A)$ is algebraic, then $\mathrm{Hom}_{\mathbb{Z}}(B_{\mathbb{Z}}^{p,p}, J^p(A))$ can be identified with set of one motives of the type

$$u : B_{\mathbb{Z}}^{p,p} \to J^p(A).$$

Definition 2.9 (Formal-p-Jacobian)**.** Let (H, V) be a formal Hodge structure of level $\leq n$. Assume H_{et} is a torsion free mixed Hodge structure. For $1 \leq p \leq n$, the pth formal Jacobian of (H, V) is defined as

$$J_{\sharp}^p(H, V) := V_p / H_{\mathrm{et}},$$

where H_{et} acts on V_p via the augmentation h. By construction, there is an extension of abelian groups

$$0 \to V_p^0 \to J_{\sharp}^p(H, V) \to J^p(H, V) \to 0,$$

where we define $J^p(H, V) := J^p(H_{\mathrm{et}}) = H_{\mathbb{C}}/(F^p + H_{\mathrm{et}})$.

Note that $J_{\sharp}^p(H, V)$ is a complex Lie group if the weights of H_{et} are $\leq 2p - 1$.

Proposition 2.10. *There is an extension of abelian groups*

$$0 \to V_p^o \to Ext^1_{\mathsf{FHS}_p}(\mathbb{Z}(-p), (H, V)) \to Ext^1_{\mathsf{MHS}}(\mathbb{Z}(-p), H_{\mathrm{et}}) \to 0,$$

for any (H, V) *formal Hodge structure of level* $\leq p + 1$. *In particular, if* H_{et} *has weights* $\leq 2p - 1$, *there is an extension*

$$0 \to V_p^o \to Ext^1_{\mathsf{FHS}_p}(\mathbb{Z}(-p), (H, V)) \to J^p(H_{\mathrm{et}}) \to 0. \tag{3}$$

Proof. By 2.4, there is a surjective map

$$\gamma : \mathrm{Ext}^1_{\mathsf{FHS}_p}(\mathbb{Z}(-p), (H, V)) \to \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), H_{\mathrm{et}}).$$

Recall that $\mathbb{Z}(-p)$ is a mixed Hodge structure and here is considered as a formal Hodge structure of level $\le p$ represented by the following diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[dr] & 0 \arrow[r] & \cdots \\ 0 \arrow[r, "h^o"'] & 0 \arrow[u] \arrow[r] & \cdots . \end{tikzcd}$$

It follows directly from the definition of a morphism of formal Hodge structures that an element of $\operatorname{Ker}\gamma$ is a formal Hodge structure of the form $(H \times \mathbb{Z}(-p), H/F)$ represented by

$$\begin{tikzcd} H_{\mathrm{et}} \times \mathbb{Z} \arrow[r] \arrow[dr, "h'_{\mathrm{et}}"] & H_{\mathbb{C}}/F^n \arrow[r] & H_{\mathbb{C}}/F^{n-1} \arrow[r] & \cdots \arrow[r] & H_{\mathbb{C}}/F^1 \\ H^o \arrow[r, "h^o"'] & V_n \arrow[u, "\pi_n"] \arrow[r, "v_n"'] & V_{n-1} \arrow[u, "\pi_{n-1}"] \arrow[r, "v_{n-1}"'] & \cdots \arrow[r] & V_1 , \arrow[u, "\pi_1"] \end{tikzcd}$$

where the augmentation $h'_{\mathrm{et}}(x, z) = h_{\mathrm{et}}(x) + \theta(z)$ for some $\theta : \mathbb{Z} \to V_p^o$. The map θ does not depend on the representative of the class of the extension because V_p and $\mathbb{Z}(-p)$ are fixed. □

Example 2.11. By the previous proposition for $p = 1$, we get

$$0 \to V_1^o \to \operatorname{Ext}^1_{\mathsf{FHS}_1}(\mathbb{Z}(-1), (H, V)) \to \operatorname{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-1), H_{\mathrm{et}}) \to 0.$$

3. SHARP COHOMOLOGY

Definition 3.1. Let X be a proper scheme over $\mathbb{C}$, $n > 0$, and $1 \le k \le n$. We define the *sharp cohomology object* $\mathrm{H}_\sharp^{n,k}(X)$ to be the n-formal Hodge structure represented by the diagram

$$\begin{tikzcd} \mathrm{H}^n(X) \arrow[r] \arrow[dr] & \mathrm{H}^n(X)_{\mathbb{C}}/F^n \arrow[r] & \cdots \arrow[r] & \mathrm{H}^n(X)_{\mathbb{C}}/F^1 \\ & V_n^{n,k}(X) \arrow[u] \arrow[r] & \cdots \arrow[r] & V_1^{n,k}(X) , \arrow[u] \end{tikzcd}$$

where

$$V_i^{n,k}(X) := \begin{cases} \mathrm{H}^{n,i}_{\mathrm{dR}}(X) & \text{if } 1 \le i \le k \\ \mathrm{H}^n(X)_{\mathbb{C}}/F^i \times_{\mathrm{H}^n(X)_{\mathbb{C}}/F^k} \mathrm{H}^{n,k}_{\mathrm{dR}}(X) & \text{if } k < i \le n. \end{cases}$$

In the case $n = k$, we will simply write $\mathrm{H}_\sharp^n(X) = \mathrm{H}_\sharp^{n,n}(X)$. This object is represented explicitly by

$$\begin{tikzcd} \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{Z}) \arrow[r] \arrow[dr] & \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{C})/F^n \arrow[r] & \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{C})/F^{n-1} \arrow[r] & \cdots \arrow[r] & \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{C})/F^1 \\ & \mathrm{H}^n(X_{\mathrm{an}}, \Omega^{<n}) \arrow[u] \arrow[r] & \mathrm{H}^n(X_{\mathrm{an}}, \Omega^{<n-1}) \arrow[u] \arrow[r] & \cdots \arrow[r] & \mathrm{H}^n(X_{\mathrm{an}}, \mathcal{O}) . \arrow[u] \end{tikzcd}$$

Example 3.2. Let X be a proper scheme of dimension d (over $\mathbb{C}$). Then $\mathrm{H}^{2d-1}(X)$ is a mixed Hodge structure satisfying $F^{d+1} = 0$, and the sharp cohomology object $\mathrm{H}_\sharp^{2d-1,d}(X)$ is represented by

$$\begin{tikzcd}
\mathrm{H}^{2d-1}(X) \arrow[r] \arrow[dr] & \mathrm{H}^{2d-1}(X)_\mathbb{C} \arrow[r, "\mathrm{id}"] & \cdots \mathrm{H}^{2d-1}(X)_\mathbb{C} \arrow[r] & \mathrm{H}^{2d-1}(X)_\mathbb{C}/F^d \cdots \\
& V_n^{2d-1,k}(X) \arrow[u] \arrow[r, "\mathrm{id}"] & \cdots V_{k+1}^{2d-1,k}(X) \arrow[u] \arrow[r] & \mathrm{H}_{\mathrm{dR}}^{2d-1,d}(X) \cdots \arrow[u]
\end{tikzcd}$$

and

$$F^{d+1}\mathrm{H}^{2d-1}(X)_\mathbb{C} \subset V_n^{2d-1,k}(X) = V_{n-1}^{2d-1,k}(X) = \cdots = V_{k+1}^{2d-1,k}(X).$$

Hence, according to Proposition 1.19, $\mathrm{H}_\sharp^{2d-1,d}(X)$ can be viewed as a formal Hodge structure of level $\leq d$.

Proposition 3.3. *For any n and $1 \leq p \leq n$, the association $X \mapsto \mathrm{H}_\sharp^{n,p}(X)$ induces a contravariant functor from the category of proper complex algebraic schemes to the category* FHS_n.

Proof. It is enough to prove the claim for $p = n$. We know that $\mathrm{H}^n(X) := \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{Z})$ along with its mixed Hodge structures is functorial in X, so for any $f : X \to Y$ we have $\mathrm{H}^n(f) : \mathrm{H}^n(Y) \to \mathrm{H}^n(X)$. Also by the theory of Kähler differentials, there exist a map of complexes of sheaves over X, $\phi_\bullet : f^*\Omega_Y^\bullet \to \Omega_X^\bullet$, inducing

$$\alpha : \mathrm{H}^n(X, f^*\Omega_Y^{<r}) \longrightarrow \mathrm{H}^n(X, \Omega_X^{<r}).$$

Moreover, there exists $\beta : \mathrm{H}^n(Y, \Omega_Y^{<r}) \to \mathrm{H}^n(X, f^*\Omega_Y^{<r})$. For it is sufficient to construct a map $\beta' : \mathrm{H}^n(Y, \Omega_Y^{<r}) \to \mathrm{H}^n(X, f^{-1}\Omega_Y^{<r})$. So let $I^\bullet$ (resp., $J^\bullet$) an injective resolution[3] of $\Omega_Y^{<r}$ (resp., $f^{-1}\Omega_Y^{<r}$). Using that f^{-1} preserves quasi-isomorphisms, we have the commutative diagram

$$\begin{tikzcd}
f^{-1}\Omega_Y^{<r} \arrow[r, "quis"] \arrow[d, "quis"] & J^\bullet \\
f^{-1}I^\bullet \arrow[ur, "\exists\gamma"'] &
\end{tikzcd},$$

where the existence of γ follows from the fact that $J^\bullet$ is injective. So we have defined a map $\psi_r : \mathrm{H}^n(Y, \Omega^{<r}) \to \mathrm{H}^n(X, \Omega^{<r})$.

[3]By injective resolution of a complex of sheaves $A^\bullet$ we mean a quasi isomorphism $A^\bullet \to I^\bullet$, where $I^\bullet$ is a complex of injective objects.

Now choosing $I_r^\bullet, J_r^\bullet$ for any r, it is easy to see that the maps ψ_r fit in the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^n(Y,\Omega^{<r}) \arrow[r] \arrow[d, "\psi_r"] & \mathrm{H}^n(Y,\Omega^{<r-1}) \arrow[r] \arrow[d, "\psi_{r-1}"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^n(X,\Omega^{<r}) \arrow[r] & \mathrm{H}^n(X,\Omega^{r-1}) \arrow[r] & \cdots .
\end{tikzcd}$$

Now it is straightforward to check that $\mathrm{H}_\sharp^{n,n}(g \circ f) = \mathrm{H}_\sharp^{n,n}(f) \circ \mathrm{H}_\sharp^{n,n}(g)$, for any $f : X \to Y$, $g : Y \to Z$. $\square$

Example 3.4 (No Künneth). Let X, Y be complete, connected, and complex varieties. Then by Künneth formula it follows that

$$\mathrm{H}^1((X \times Y)_{\mathrm{an}}, ?) = \mathrm{H}^1(X_{\mathrm{an}}, ?) \oplus \mathrm{H}^1(Y_{\mathrm{an}}, ?) \qquad ? = \mathbb{Z},$$

so that $\mathrm{H}_\sharp^1(X \times Y) = \mathrm{H}_\sharp^1(X) \oplus \mathrm{H}_\sharp^1(Y)$. But as soon as we move in degree 2, there is no hope for a good formula. With the same notation, we get

$$\mathrm{H}^2((X \times Y))_{\mathbb{Q}} = \mathrm{H}^2(X)_{\mathbb{Q}} \oplus \mathrm{H}^1(X)_{\mathbb{Q}} \otimes \mathrm{H}^1(Y)_{\mathbb{Q}} \oplus \mathrm{H}^2(Y)_{\mathbb{Q}},$$

which is the usual decomposition of singular cohomology. Let $p : X \times Y \to X$, $q : X \times Y \to Y$ the two projections; note that

$$\mathcal{O}_{X\times Y} \to \Omega^1_{X\times Y} = \sigma^{<2}\big(p^*(\mathcal{O}_X \to \Omega^1_X) \otimes q^*(\mathcal{O}_Y \to \Omega^1_Y)\big);$$

hence, there is a canonical map

$$\mathrm{H}^2(X \times Y, p^*(\Omega_X^{<2}) \otimes q^*(\Omega_Y^{<2})) = \oplus_{i=0}^2 \mathrm{H}_{\mathrm{dR}}^{2-i,2}(X) \otimes \mathrm{H}_{\mathrm{dR}}^{i,2}(Y) \to \mathrm{H}_{\mathrm{dR}}^{2,2}(X \times Y),$$

which is not necessarily an isomorphism. From this it follows that we cannot have a Künneth formula for $\mathrm{H}_\sharp^{2,2}(X \times Y)$.

3.1. The Generalized Albanese of Esnault–Srinivas–Viehweg

Let X be a proper and irreducible algebraic scheme of dimension d over $\mathbb{C}$. Then there exists an algebraic group, say $\mathrm{ESV}(X)$, such that $\mathrm{ESV}(X)_{\mathrm{an}} = \mathrm{H}^{2d-1}(X, \Omega^{<d})/\mathrm{H}^{2d-1}(X_{\mathrm{an}}, \mathbb{Z})$ and it fits in the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} c \arrow[r] \arrow[d, "\rho"] & \frac{\mathrm{H}^{2d-1}(X)_{\mathbb{C}}}{\mathrm{H}^{2d-1}(X)} \arrow[r, "c"] \arrow[d, "\alpha"] & J^d(\mathrm{H}^{2d-1}(X)) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \operatorname{Ker} \theta \arrow[r] & \frac{\mathrm{H}^{2d-1,d}_{\mathrm{dR}}(X)}{\mathrm{H}^{2d-1}(X)} \arrow[r, "\theta"] & J^d(\mathrm{H}^{2d-1}(X)) \arrow[r] & 0 ,
\end{tikzcd}$$

where α is induced by de canonical map of complexes of analytic sheaves $\mathbb{C} \to \Omega^{<d}$. (See [6, Theorem 1, Lemma 3.1].)

Recall that the formal Hodge structure (of level $\leq 2d-1$) $\mathrm{H}_{\sharp}^{2d-1,d}(X)$ can be viewed as a fhs of level $\leq d$ (see 3.2) represented by the following diagram:

$$\begin{tikzcd}
\mathrm{H}^{2d-1}(X) \arrow[r] \arrow[dr, "h"] & \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^{d} \arrow[r] & \cdots \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^{1} \\
 & \mathrm{H}_{\mathrm{dR}}^{2d-1,d}(X) \arrow[u] \arrow[r] & \cdots \mathrm{H}_{\mathrm{dR}}^{2d-1,1}(X)\,. \arrow[u]
\end{tikzcd}$$

Proposition 3.5. *There is an isomorphism of complex connected Lie groups* (*not only of abelian groups*!)

$$\mathrm{ESV}(X)_{\mathrm{an}} \cong Ext^{1}_{\mathsf{FHS}_{\mathsf{d}}}(\mathbb{Z}(-d), \mathrm{H}_{\sharp}^{2d-1,d}(X)),$$

where $\mathbb{Z}(-d)$ is the Tate structure of type (d,d) viewed as an étale formal Hodge structure.

Proof. **Step 1.** By [2], there is a canonical isomorphism of Lie groups

$$\mathrm{ESV}_{\mathrm{an}}(X) \cong \mathrm{Ext}^{1}_{{}^{t}\mathcal{M}_1^{\mathrm{a}}}([\mathbb{Z} \to 0], [0 \to \mathrm{ESV}(X)]) \cong \mathrm{Ext}^{1}_{\mathsf{FHS}_1(1)}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X)))$$

(recall that in [2] $\mathsf{FHS}_1(1)$ is simply denoted by FHS_1; ${}^{t}\mathcal{M}_1^{\mathrm{a}}$ is the category of generalized 1-motives with torsion), where $T_{\oint}(\mathrm{ESV}(X))$ is the formal Hodge structure represented by

$$\begin{tikzcd}
\mathrm{H}^{2d-1}(X)(d) \arrow[r] \arrow[dr] & \mathrm{H}^{2d-1}(X)_{\mathbb{C}}(d)/F^{0} \\
 & \mathrm{H}_{\mathrm{dR}}^{2d-1,d}(X)\,. \arrow[u]
\end{tikzcd}$$

Step 2. Up to a twist by $-d$, we can view $T_{\oint}(\mathrm{ESV}(X))$ as an object of FHS_d, say (H_{et}, V) with $H_{\mathrm{et}} = \mathrm{H}^{2d-1}(X)$, $V_d = \mathrm{H}_{\mathrm{dR}}^{2d-1,d}(X)$, $V_i = 0$ for $1 \leq i < d$. It is easy to check that $\mathrm{Ext}^{1}_{\mathsf{FHS}_1(1)}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X))) = \mathrm{Ext}^{1}_{\mathsf{FHS}_d}(\mathbb{Z}(-d), (H_{\mathrm{et}}, V))$. Then applying $\mathrm{Ext}^{1}_{\mathsf{FHS}_d}(\mathbb{Z}(-d), -)$ to the canonical inclusion $(H_{\mathrm{et}}, V) \subset \mathrm{H}_{\sharp}^{2d-1,d}(X)$, we get a natural map

$$\mathrm{Ext}^{1}_{\mathsf{FHS}_1(1)}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X))) \to \mathrm{Ext}^{1}_{\mathsf{FHS}_d}(\mathbb{Z}(-d), \mathrm{H}_{\sharp}^{2d-1,d}(X)),$$

which is an isomorphism by (3). □

3.2. The Generalized Albanese of Faltings and Wüstholz

Let U be a smooth algebraic scheme over $\mathbb{C}$. Then it is possible to construct a smooth compactification, i.e., $\exists\ j: U \to X$ open embedding with X proper and smooth. Moreover, we can suppose that the complement $Y := X \setminus U$ is a normal crossing divisor.[4]

[4]It is possible to replace $\mathbb{C}$ with a field $\boldsymbol{k}$ of characteristic zero. In that case, we must assume that there exists a $\boldsymbol{k}$ rational point in order to have $\mathrm{FW}(Z)$ defined over $\boldsymbol{k}$.

Remark 3.6. There is a commutative diagram (see [8, §3])

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0(X_{\mathrm{an}}, \Omega^1(\log Y)) \arrow[r] \arrow[d, "a"] & \mathrm{H}^1(U)_{\mathbb{C}} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^{1,1}_{\mathrm{dR}}(X) \arrow[r] \arrow[d, "b"] & 0 \\ 0 \arrow[r] & \mathrm{H}^1(\Gamma(U_{an}, \Omega^\bullet)) \arrow[r] & \mathrm{H}^1(U)_{\mathbb{C}} \arrow[r] & \mathrm{H}^{1,1}_{\mathrm{dR}}(U) . \end{tikzcd}$$

Hence, by the snake lemma, $\operatorname{Ker} b \cong \operatorname{Coker} a$. We identify these two $\mathbb{C}$-vector spaces, and we denote both by K.

For any $Z \subset K$ subvector space, we define the $\mathbb{C}$-linear map $\alpha_Z : \mathrm{H}^1(X, \mathcal{O})^* \to Z^*$ as the dual of the canonical inclusion $Z \subset \mathrm{H}^1(X, \mathcal{O})$.

Defintion 3.7 (The Generalized Albanese of Serre)**.** We know that

$$\mathrm{H}^1(U)(1) = T_{Hodge}([\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)])$$

and that the generalized Albanese of Serre is the Cartier dual of the above 1-motive, i.e.,

$$[0 \to \mathrm{Ser}(U)] = [\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)]^\vee.$$

Note that by construction $\mathrm{Ser}(U)$ is a semi-abelian group scheme corresponding to the mixed Hodge structure $\mathrm{H}^1(U)(1)^\vee := \mathcal{H}om_{\mathsf{MHS}}(\mathrm{H}^1(U)(1), \mathbb{Z}(1))$.

The universal vector extension of $\mathrm{Ser}(U)$ is

$$0 \to \underline{\omega}_{\mathrm{Pic}^0(X)} \to \mathrm{Ser}(U)^\natural \to \mathrm{Ser}(U) \to 0$$

this follows by the construction of $\mathrm{Ser}(U)$ as the Cartier dual of $[\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)]$ and [3] Lemma 2.2.4.

Recall that $\mathrm{Lie}(\mathrm{Pic}^0(X)) = \mathrm{H}^1(X, \mathcal{O})$, then $\underline{\omega}_{\mathrm{Pic}^0(X)}(\mathbb{C}) = \mathrm{H}^1(X, \mathcal{O})^*$.

Defintion 3.8 (The Gen. Albanese of Faltings and Wüstholz)**.** We define an algebraic group $\mathrm{FW}(Z)$ (depending on U and the choice of the vector space Z) to be the vector extension of $\mathrm{Ser}(U)$ by Z^* defined by

$$\alpha_Z \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^1(X, \mathcal{O})^*, Z^*) \cong \mathrm{Hom}_{\mathbb{C}}(\omega_{\mathrm{Pic}^0(X)}, Z^*) \cong \mathrm{Ext}^1(\mathrm{Ser}(U), Z^*),$$

i.e., $\mathrm{FW}(Z)$ is the push-forward

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(X, \mathcal{O})^* \arrow[r] \arrow[d, "\alpha_Z"] & \mathrm{Ser}(U)^\natural \arrow[r] \arrow[d] & \mathrm{Ser}(U) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & Z^* \arrow[r] & \mathrm{FW}(Z) \arrow[r] & \mathrm{Ser}(U) \arrow[r] & 0 . \end{tikzcd}$$

Proposition 3.9. *With the above notation consider the formal Hodge structure* $(H_{\mathrm{et}}, V) \in \mathsf{FHS}_1$ *represented by*

$$\begin{tikzcd} \mathrm{H}^1(U)(1)^\vee \arrow[r] \arrow[dr, "h"'] & \mathrm{H}^0(X_{\mathrm{an}}, \Omega^1(\log Y))^* \\ & \mathrm{H}^1(\Gamma(U_{\mathrm{an}}, \Omega^\bullet))^* . \arrow[u, "a^*"'] \end{tikzcd}$$

(*This diagram is the dual of the left square in Remark* 3.6). *Recall that* $K = Ker\, a$. *Then*

$$\mathrm{FW}(K)_{\mathrm{an}} \cong Ext^1_{\mathsf{FHS}_1}(\mathbb{Z}(-1), (H_{\mathrm{et}}, V)).$$

Proof. It is a direct consequence of 2.10. □

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