Canadian Mathematical Bulletin
http://dx.doi.org/10.4153/CMB-2017-076-2

On Deformations of 1-motives

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Abstract. According to a well-known theorem of Serre and Tate, the infinitesimal deformation theory of an abelian variety in positive characteristic is equivalent to the infinitesimal deformation theory of its Barsotti–Tate group. We extend this result to 1-motives.

1 Introduction

Let R be an artinian local ring with maximal ideal $\mathfrak{m}$ and perfect residue field k of positive characteristic p. Let $\mathcal{M}_1(R)$ denote the category of (smooth) 1-motives over R. For any 1-motive M (resp. Barsotti–Tate group $\mathcal{B}$) over R, let M_0 (resp. $\mathcal{B}_0$) denote its base change to $k = R/\mathfrak{m}$. Let $\mathrm{Def}(R,k)$ denote the category whose objects are triples $(M_0, \mathcal{B}, \varepsilon_0)$ where M_0 is a 1-motive over k, $\mathcal{B}$ is a Barsotti–Tate group over R and $\varepsilon_0 : \mathcal{B}_0 \to M_0[p^\infty]$ is an isomorphism of $\mathcal{B}_0$ with the Barsotti–Tate group of M_0. The aim of this paper is to prove the following theorem.

Theorem 1.1 *The functor*

$$\begin{aligned} \Delta_R : \mathcal{M}_1(R) &\longrightarrow \mathrm{Def}(R,k) \\ M &\longmapsto (M_0, M[p^\infty], \textit{ natural } \varepsilon_0), \end{aligned} \tag{1.1}$$

is an equivalence of categories.

This result generalises the well-known Serre–Tate equivalence for abelian schemes over artinian local rings R as above (*cf.* [6, Theorem 1.2.1], [8, V, Theorem 2.3]). Arguments in [6,8] do not extend directly to the case of 1-motives but are used to get some intermediate results. Note that the proof of fullness (Proposition 4.3) and essential surjectivity (Proposition 4.5) uses weights, a careful study of the case of 1-motives of the form $[\mathbb{Z} \to \mathbb{G}_m]$ (Lemma 2.4), which is orthogonal to the classical case of abelian schemes, and Galois descent arguments (Lemma 4.4). Another essential ingredient is that tori and étale group schemes over k lift uniquely to R.

As an application, in Section 4.3 we extend the description of the formal moduli space of an ordinary abelian variety over an algebraically closed field and its Serre–Tate coordinates to 1-motives. Consequently, canonical liftings of (ordinary) 1-motives over k to 1-motives over $W(k)$ exist (Proposition 4.10).

Received by the editors October 10, 2017; revised November 17, 2017.
Published electronically March 22, 2018.
AMS subject classification: **14L15**, 14C15, 14L05.
Keywords: 1-motive, Barsotti-Tate group.

Notation

Let $p^s, s \geq 1$, denote the characteristic of R. Then R is canonically endowed with the structure of a finite $W_s(k)$-algebra, where $W_s(k) = W(k)/(p^s)$ denotes the ring of Witt vectors of length s. We also fix a positive integer n such that $(1+\mathfrak{m})^{p^n} = \{1\}$. For any R-group scheme G, we identify $G(R) = \mathrm{Hom}_R(\mathrm{Spec}\, R, G)$ with $\mathrm{Hom}_{R\text{-gr}}(\mathbb{Z}, G)$ by mapping $a \in G(R)$ to the morphism $u\colon \mathbb{Z} \to G$ such that $u(1) = a$.

2 Barsotti–Tate Groups and 1-motives

Let $\mathcal{M}_1(R)$ be the category of (smooth) 1-motives over R. Its objects are two term complexes of commutative R-group schemes $M = [u\colon L \to G]$ in degrees -1 and 0, where G is extension of an abelian scheme A by a torus T, and L is locally for the étale topology on R isomorphic to $\mathbb{Z}^r$ for some non-negative integer r. Recall that a 1-motive has a natural weight filtration

$$0 \subseteq W_{-2}M = [0 \to T] \subseteq W_{-1}M = [0 \to G] \subseteq W_0 M = M.$$

Since morphisms of 1-motives respect filtrations, $\mathcal{M}_1(R)$ is a filtered category. We will denote by $\mathcal{M}_1(R)_{\leq i}$ the full subcategory of $\mathcal{M}_1(R)$ consisting of 1-motives such that $M = W_i M$. Given a 1-motive M, let $M_{\mathrm{ab}} = [u_{\mathrm{ab}}\colon L \to A]$, where u_{ab} is the composition of u with the canonical morphism $G \to A$. Let $\mathcal{M}_1(R)_{\geq -1}$ be the full subcategory of $\mathcal{M}_1(R)$ consisting of 1-motives $M = M_{\mathrm{ab}}$. Similarly $\mathrm{Def}(R,k)_{\geq -1}$ (resp. $\mathrm{Def}(R,k)_{\leq -1}$) denotes the full subcategory of $\mathrm{Def}(R,k)$ consisting of objects such that M_0 is in $\mathcal{M}_1(k)_{\geq -1}$ (resp. in $\mathcal{M}_1(k)_{\leq -1}$).

For any $m \in \mathbb{N}$, consider the cone of the multiplication by m on M:

$$M/mM\colon L \xrightarrow{\binom{-u}{-m}} G \oplus L \xrightarrow{(-m,u)} G,$$

where L is in degree -2, and let $M[m] = H^{-1}(M/mM)$ (see [4, 10.1.4] or [2, §1.3] with a different sign convention); it is a finite and flat R-group scheme and it fits in the diagram

(2.1)
$$\begin{array}{lccccccccc}
 & & & & L & = & L & & \\
 & & & & \downarrow \binom{-u}{-m} & & \downarrow -m & & \\
\widetilde{\eta}_{M[m]}\colon & 0 & \to & G[m] & \to & \mathrm{Ker}((-m,u)) & \to & L & \to & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \\
\eta_{M[m]}\colon & 0 & \to & G[m] & \to & M[m] & \to & L/mL & \to & 0.
\end{array}$$

Remark 2.1 Let us consider

$$\xi_{G[m]}\colon 0 \longrightarrow G[m] \longrightarrow G \xrightarrow{m} G \longrightarrow 0.$$

By direct computations one gets $\widetilde{\eta}_{M[m]} = -u^* \xi_{G[m]}$ as elements of $\mathrm{Ext}_R(L, G[m])$.

Since the push-out of the extension $\widetilde{\eta}_{M[p^r]}$ along the canonical morphism $G[p^r] \to G[p^{r+1}]$ is isomorphic to $p \cdot \widetilde{\eta}_{M[p^{r+1}]}$, there is a morphism of complexes

$\eta_{M[p^r]} \to \eta_{M[p^{r+1}]}$ for any $r \geq 1$. One then gets a short exact sequence of Barsotti–Tate groups (BT groups for short)

$$\eta_{M[p^\infty]}\colon 0 \longrightarrow G[p^\infty] \longrightarrow M[p^\infty] \longrightarrow L[p^\infty] \longrightarrow 0, \tag{2.2}$$

where $L[p^\infty] := L \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p$.

Lemma 2.2 *Let $M_0 = [u_0\colon L_0 \to G_0]$, let M_0' be* 1*-motives over k, and let $\varphi_0\colon M_0 \to M_0'$ be a morphism of* 1*-motives. Denote by T_0 the maximal subtorus of G_0.*

(i) *Any finite and flat R-group scheme F that lifts $M_0[m]$ is endowed with a unique weight filtration that lifts the weight filtration $T_0[m] \subseteq G_0[m] \subseteq M_0[m]$.*
(ii) *Any morphism of finite flat R-group schemes $F \to F'$ that lifts $\varphi_0[m]$ respects filtrations.*
(iii) *Any BT group over R that lifts $M_0[p^\infty]$ admits a unique filtration that lifts the weight filtration on $M_0[p^\infty]$ and any morphism of BT groups over R which lifts $\varphi_0[p^\infty]$ respects filtrations.*
(iv) *If M_0 lifts to a* 1*-motive M over R, then the natural weight filtration on $M[m]$ agrees with the one induced by the weight filtration on M_0.*

Proof Let F be a finite flat R-group scheme lifting $M_0[m]$. Let L be the unique étale lifting of L_0 over R. We claim that F fits into a short exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow L/mL \longrightarrow 0,$$

which lifts the sequence $\eta_{M_0[m]}$ for M_0 in diagram (2.1). Indeed, there exists a unique lifting $f\colon F \to L/mL$ of the morphism $M_0[m] \to L_0/mL_0$, since L/mL is étale, and we set $E = \operatorname{Ker} f$. Note further that f factors through the epimorphism $\pi_F\colon F \to F^{\text{ét}} = F/F^\circ$ with F° the identity component of F. Let $f^{\text{ét}}\colon F^{\text{ét}} \to L/mL$ satisfy $f^{\text{ét}} \circ \pi_F = f$. The morphism $f_0^{\text{ét}}$ is an epimorphism of finite étale k-group schemes, hence so is $f^{\text{ét}}$. It follows that f is an fppf epimorphism, hence the short exact sequence as claimed.

Let L_0^* and F^* be the Cartier duals of L_0 and F, respectively, and let T be the unique torus lifting T_0. We have seen that the canonical morphism $M_0[m]^* \to L_0^*/mL_0^*$ lifts uniquely to a morphism $F^* \to L^*/mL^*$ over R and hence, by Cartier duality, the immersion $T_0[m] \to M_0[m]$ lifts uniquely to an immersion $T[m] \to F$ over R. The composition $T[m] \to F \to L/mL$ is the zero map, since its reduction modulo $\mathfrak{m}$ is the zero map and L/mL is étale. The closed immersion $T[m] \to F$ factors thus through E. By construction, the filtration

$$0 \subseteq T[m] \subseteq E \subseteq F$$

of F lifts the weight filtration on $M_0[m]$, so the existence of a filtration as in (i) is proved.

With similar arguments, one shows statement (ii). Statement (iv) and the uniqueness of the filtration in (i) follow from (ii), and statement (iii) is a formal consequence of (i) and (ii). ∎

The classical Serre–Tate theorem states that deformations of an abelian variety over k only depend on deformations of its BT group. As we will now explain, the analogous result for 1-motives of the type $[\mathbb{Z} \to \mathbb{G}_{m,k}]$ can explicitly be deduced from the canonical isomorphism

$$R^* \xrightarrow{\sim} k^* \times (1+\mathfrak{m}), \quad x \longmapsto \big(x_0, x/[x_0]\big), \tag{2.3}$$

where x_0 is the reduction of x modulo $\mathfrak{m}$ and $[x_0] \in W_s(k)$ is the multiplicative representative of x_0. Recall that $(1+\mathfrak{m})^{p^n} = \{1\}$, and hence $1+\mathfrak{m} = \mu_{p^n}(R) = \mu_{p^\infty}(R)$.

We now consider BT groups that are extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by μ_{p^∞}. By [8, Proposition 2.5, p. 180], the map

$$\Psi: 1+\mathfrak{m} \longrightarrow \mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}),$$

which associates with $u \in 1+\mathfrak{m}$ the push-out along $u: \mathbb{Z} \to \mu_{p^\infty}$ of the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

is an isomorphism. In particular, p^n kills $\mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$. For any $u \in R^*$, let $M^u = [u: \mathbb{Z} \to \mathbb{G}_{m,R}]$ and consider the homomorphism

$$\Phi: 1+\mathfrak{m} \longrightarrow \mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}), \tag{2.4}$$

which maps $u \in 1+\mathfrak{m} \subset R^*$ to the extension $\eta_{M^u[p^\infty]}$ as in (2.2).

Lemma 2.3 *We have $\Phi = -\Psi$; hence, Φ is an isomorphism.*

Proof Let $\mathrm{Ext}_{\mathbb{Z}/p^n\mathbb{Z}}(\cdot,\cdot)$ denote classes of extensions of p^n-torsion R-group schemes. By [8, p. 183], we have isomorphisms

$$\begin{aligned} \mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) &\simeq \varprojlim_m \mathrm{Ext}_R(\mathbb{Z}/p^m\mathbb{Z}, \mu_{p^\infty}) \\ &\simeq \mathrm{Ext}_R(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^\infty}) \simeq \mathrm{Ext}_{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}), \end{aligned} \tag{2.5}$$

where the first isomorphism is induced by pull-back along $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p$, the second isomorphism follows, since p^n kills $\mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$, and the third isomorphism is obtained by passing to kernels of the multiplication by p^n, since μ_{p^∞} is divisible. Note that the composition of the isomorphisms in (2.5) associates with an extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by μ_{p^∞} the corresponding sequence of kernels of the multiplication by p^n. Let

$$\Psi_n, \Phi_n : 1+\mathfrak{m} \longrightarrow \mathrm{Ext}_{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n})$$

denote the composition of Ψ (resp. Φ) with the isomorphisms in (2.5). We are left to prove that $\Phi_n = -\Psi_n$.

Note that $\Psi_n(u)$ can also be constructed taking first the push-out along $u: \mathbb{Z} \to \mathbb{G}_{m,R}$ of $\zeta: 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ and then considering the sequence of kernels of the multiplication by p^n. Indeed, $u_*\zeta = \iota_n^*\iota_*\Psi(u)$ with $\iota_n: \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p$ and $\iota: \mu_{p^\infty} \to \mathbb{G}_{m,R}$ the usual morphisms. Since Cartier duality induces the identity both on $1+\mathfrak{m} = \mathrm{Hom}_{R\text{-gr}}(\mathbb{Z}, \mathbb{G}_m)$ and on $\mathrm{Ext}_{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n})$, $\Psi_n(u)$ is also represented by the sequence of cokernels of the multiplication by p^n of the sequence obtained via pull-back along $u: \mathbb{Z} \to \mathbb{G}_{m,R}$ from the sequence $\xi_{\mu_{p^n}}: 0 \to \mu_{p^n} \to \mathbb{G}_{m,R} \to \mathbb{G}_{m,R} \to 0$. On the other hand, $\Phi_n(u)$ is the sequence $\eta_{M^u[p^n]}$ in diagram (2.1). By Remark 2.1 and diagram (2.1), we conclude that $\Phi_n(u) = -\Psi_n(u)$. ■

We now see how to interpret (2.3) in terms of deformations of 1-motives.

Lemma 2.4 *Given a* 1*-motive* $M_0 = [u_0\colon \mathbb{Z} \to \mathbb{G}_{m,k}]$ *and a* $\mathcal{B} \in \mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$ *there is a unique* 1*-motive* $M = [u\colon \mathbb{Z} \to \mathbb{G}_{m,R}]$ *that lifts* M_0 *and whose BT group is isomorphic to* $\mathcal{B}$ *as extension of* μ_{p^∞} *by* $\mathbb{Q}_p/\mathbb{Z}_p$*.*

Proof Note that if $\mathcal{B}, \mathcal{B}'$ are two liftings of $M_0[p^\infty]$, there is at most one morphism $\mathcal{B} \to \mathcal{B}'$ as extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by μ_{p^∞}. We have to prove that the homomorphism

$$R^* \longrightarrow k^* \times \mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}), \quad u \longmapsto (u_0, \eta_{M^u[p^\infty]})$$

is an isomorphism. Note that if $u = [u_0], u_0 \in k^*$, then u admits p^r-th roots for any $r \geq 1$, since k is perfect; hence, $\eta_{M^u[p^\infty]}$ is split, since $\mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$ is killed by p^n. We are then reduced to checking that the homomorphism

$$1 + \mathfrak{m} \longrightarrow \mathrm{Ext}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}), \quad u \longmapsto \eta_{M^u[p^\infty]},$$

is an isomorphism. But this homomorphism equals Φ in (2.4), and the latter is an isomorphism by Lemma 2.3. ∎

3 Auxiliary Results

3.1 Galois Actions

Let k'/k be a finite Galois extension of $\Gamma = \mathrm{Gal}(k'/k)$ and set $R' = R \otimes_{W(k)} W(k')$. Note that R' is an artinian local ring with residue field k' and $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ is a Galois covering of group Γ. Then Γ naturally acts on $\mathcal{M}_1(R')$ and $\mathrm{Def}(R', k')$ and the Serre–Tate functor $\Delta_{R'}$ (1.1) commutes with the Galois action.

Note that the datum of a 1-motive M over R is equivalent to the datum of a 1-motive $M' = [u'\colon L' \to G']$ over R' with a Γ-action compatible with the Γ-action on $\mathrm{Spec}\, R'$. Indeed, L' descends to a lattice over R. Further, since the topological space underlying G' coincides with the topological space underlying the semi-abelian k'-variety G'_0, it can be covered by affine open subschemes which are stable under the Γ-action and hence it descends to an R-group scheme. The maximal subtorus T' of G' descends to a subtorus T of G and G/T is an abelian scheme, since it is an abelian scheme after base change to R'. Similarly, the datum of an object $(G_0, \mathcal{B}, \varepsilon_0)$ in $\mathrm{Def}(R, k)$ is equivalent to the datum of a $(G'_0, \mathcal{B}', \varepsilon'_0)$ in $\mathrm{Def}(R', k')$ together with a Γ-action compatible with the Γ-action on k' (in the first and third entries) and the Γ-action on R' (on the second entry).

3.2 Refinements

Recall that any 1-motive $M = [u\colon L \to G]$ over R admits a so-called universal extension $M^\natural = [u^\natural\colon L \to G^\natural]$ that fits into a short exact sequence of complexes of R-group schemes

$$0 \longrightarrow [0 \longrightarrow \mathbb{V}(M)] \longrightarrow M^\natural \longrightarrow M \longrightarrow 0, \tag{3.1}$$

where $\mathbb{V}(M)$ is the vector group associated to the dual sheaf of $\underline{\mathrm{Ext}}^1_{\mathrm{Zar}}(M, \mathbb{G}_{a,R})$. Note that $\mathbb{V}(M)$ is killed by p^s, since the multiplication by p^s morphism is the 0 map on $\mathbb{G}_{a,R}$. Note that $M^\natural$ is denoted by $\mathbb{E}(M) = [L \to \mathbb{E}(M)_G]$ in [1, 2].

Remark 3.1 We can refine some preliminary results in [6]. Let $s' \geq s$ be the minimal integer such that $\mathfrak{m}^{s'} = 0$.

(i) Let G be any smooth commutative R-group scheme. By [6, Lemma 1.1.2] the kernel of the reduction map $G(R) \to G(k)$ is killed by $p^{s(s'-1)}$. In our hypothesis one can prove that it is killed by $p^{s'-1}$. Indeed, by the theory of Greenberg functor the sections $G(R/\mathfrak{m}^i)$, $1 \leq i \leq s'$, can be identified with the group of k-rational sections of a smooth k-group scheme $\mathrm{Gr}_i(G)$. Further, there are so-called change of level morphisms $\rho_i^1\colon \mathrm{Gr}_{i+1}(G) \to \mathrm{Gr}_i(G)$ such that $\rho_i^1(k)$ coincides with the reduction map $G(R/\mathfrak{m}^{i+1}) \to G(R/\mathfrak{m}^i)$. By Greenberg's structure theorem ([5],[3, Thm. A.17]) the kernel of ρ_i^1 is a k-vector group, thus killed by p. It follows then by induction that $\mathrm{Ker}(G(R) \to G(k))$ is killed by $p^{s'-1}$.

(ii) By [6, Lemma 1.1.3 (3)] a morphism of semi-abelian varieties $\varphi_0\colon G_0 \to H_0$ lifts over R up to multiplication by $p^{s(s'-1)}$. We can also improve this estimate if $p > 2$. Let M, N be 1-motives over R. By [1, Thm. 2.1] there exists a canonical morphism between universal extensions (3.1) $\varphi^\natural\colon M^\natural \to N^\natural$ that lifts $\varphi_0^\natural$. If $\varphi^\natural(\mathbb{V}(M)) \subseteq \mathbb{V}(N)$, then $\varphi^\natural$ induces a morphism $\varphi\colon M \to N$ that lifts φ_0. In general, since the multiplication by p^s morphism kills $\mathbb{V}(M)$, the morphism $p^s\varphi^\natural$ maps $\mathbb{V}(M)$ to 0. Hence, $p^s\varphi_0$ lifts to a morphism "$p^s\varphi$": $M \to N$.

4 Proof of the Main Theorem

4.1 Full Faithfulness

Proposition 4.1 *Let M, N be two 1-motives over R. Then the reduction map*

$$\mathrm{Hom}_{\mathcal{M}_1(R)}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{M}_1(k)}(M_0, N_0)$$

is injective.

Proof Consider a morphism $\varphi\colon M \to N$ and assume that its reduction modulo $\mathfrak{m}$ is the 0 morphism. If $p > 2$, then $\varphi = 0$, since the induced morphism between universal extensions $\varphi^\natural\colon M^\natural \to N^\natural$ is the zero map [1, Thm. 2.1]. In general, one has $\varphi = 0$ in degree -1 by the equivalence of the étale sites over R and over k, and $\varphi = 0$ in degree 0 by [6, Lemma 1.1.3 2)]. ■

Corollary 4.2 *The functor (1.1) is faithful.*

Proposition 4.3 *The functor (1.1) is full.*

Proof Let $M = [u\colon L \to G]$, $N = [v\colon F \to H]$ be two 1-motives over R, $\varphi_0\colon M_0 \to N_0$ a morphism between their reduction modulo $\mathfrak{m}$, and $\psi\colon M[p^\infty] \to N[p^\infty]$ a lifting of $\varphi_0[p^\infty]\colon M_0[p^\infty] \to N_0[p^\infty]$. We have to prove that there exists a lifting $\varphi\colon M \to N$ of φ_0 over R such that $\varphi[p^\infty] = \psi$.

As a first step, consider the case $L = F = 0$. We assume that $p > 2$; if $p = 2$, the same proof works replacing s with $s(s'-1)$, where $\mathfrak{m}^{s'} = 0$, and it coincides with the one in [6, p. 144]. Given a $\varphi_0 \colon G_0 \to H_0$ and a morphism $\psi \colon G[p^\infty] \to H[p^\infty]$ lifting $\varphi_0[p^\infty]$, by Remark 3.1(ii) there exists a morphism "$p^s\varphi$": $G \to H$ lifting $p^s\varphi_0$. Further, $p^s\psi =$ "$p^s\varphi$"$[p^\infty]$ by [6, Lemma 1.1.3(2)], since both morphisms lift $p^s\varphi_0[p^\infty]$. Hence, "$p^s\varphi$" kills $G[p^s]$ and thus "$p^s\varphi$" $= p^s\varphi$ for a morphism $\varphi \colon G \to H$ (necessarily unique since $\mathrm{Hom}_{R\text{-gr}}(G,H)$ has no p-torsion). Thus, the restriction of the functor (1.1) to $\mathcal{M}_1(R)_{\leq -1}$ is full.

As a second step, consider the case where G and H are abelian varieties. Fullness of the restriction of the functor (1.1) to $\mathcal{M}_1(R)_{\geq -1}$ follows from the previous step via Cartier duality.

For the general case, let $M = [u \colon L \to G]$, $N = [v \colon F \to H]$ be two 1-motives over R, $\varphi_0 \colon M_0 \to N_0$ a morphism between their reduction modulo $\mathfrak{m}$ and $\psi \colon M[p^\infty] \to N[p^\infty]$ a lifting of $\varphi_0[p^\infty] \colon M_0[p^\infty] \to N_0[p^\infty]$. Let $\varphi_0 = (f_0, g_0)$, *i.e.*, $f_0 \colon L_0 \to F_0$, $g_0 \colon G_0 \to H_0$ and $g_0 \circ u_0 = v_0 \circ f_0$. Recall that any lifting ψ of $\varphi_0[p^\infty]$ respects weight filtrations by Lemma 2.2. Hence, by the first step of the proof there exists a unique morphism $g \colon G \to H$ lifting g_0. Further, by the equivalence between the category of étale group schemes over k and the category of étale group schemes over R, there exists a unique $f \colon L \to F$ lifting f_0. We are left to prove that $g \circ u = v \circ f$, so that $\varphi = (f,g)$ is a morphism of 1-motives, and that $\varphi[p^\infty] = \psi$. The latter equality follows from [6, Lemma 1.1.3 2)], since both morphisms are liftings of $\varphi_0[p^\infty]$.

Let $Z = [w \colon L \to H]$ with $w = g \circ u - v \circ f$. We claim that $\eta_{Z[p^\infty]}$ in (2.2) is split. For proving this claim, it is sufficient to check that $\eta_{Z[p^r]}$ (and hence $\widetilde{\eta}_{Z[p^r]}$ in (2.1)) is split for any r. Since $\psi[p^r]$ restricts to the morphism $g[p^r] \colon G[p^r] \to H[p^r]$ on weight ≤ -1 subgroups and induces $f/p^r f \colon L/p^r L \to F/p^r F$ in weight 0, it is

$$(f/p^r f)^* \eta_{N[p^r]} = g[p^r]_* \eta_{M[p^r]},$$

and hence

$$f^* \widetilde{\eta}_{N[p^r]} = g[p^r]_* \widetilde{\eta}_{M[p^r]}.$$

By Remark 2.1, we have

$$\begin{aligned}\widetilde{\eta}_{Z[p^r]} &= (-w)^* \xi_{H[p^r]} = f^*(v^* \xi_{H[p^r]}) - u^*(g^* \xi_{H[p^r]}) = f^* \widetilde{\eta}_{N[p^r]} - u^* g[p^r]_* \xi_{G[p^r]} \\ &= f^* \widetilde{\eta}_{N[p^r]} - g[p^r]_* \widetilde{\eta}_{M[p^r]} = 0.\end{aligned}$$

Hence, Z is a 1-motive such that $\eta_{Z[p^\infty]}$ is split extension of $L[p^\infty]$ by $H[p^\infty]$ and $w_0 = 0$. We are left to prove that $w = 0$. We can work étale locally on R and then assume that L is constant and the maximal subtorus S of H is split. By the second step of this proof, $w_{\mathrm{ab}} \colon L \to B$ is the 0 morphism. Hence, w factors through S. Let $M_t = [w \colon L \to S]$ and note that $M_t[p^\infty]$ is split extension of $L[p^\infty]$ by $S[p^\infty]$. Hence, $w = 0$ by Lemma 2.4. ■

4.2 Essential Surjectivity

The strategy of the proof of essential surjectivity is first to construct the desired 1-motive étale locally and then to apply descent. We then need the following result.

Lemma 4.4 *Let k'/k be a finite Galois extension of Galois group Γ and set $R' = R \otimes_{W(k)} W(k')$. A 1-motive M' over R' descends to R if and only if its image in $\mathrm{Def}(R', k')$ via (1.1) descends to $\mathrm{Def}(R, k)$.*

Proof The necessity is clear. For sufficiency, let $M' = [u' : L' \to G']$ and assume that $(M'_0, \mathcal{B}' = M'[p^\infty], \mathrm{can} : \mathcal{B}'_0 \xrightarrow{\sim} M'_0[p^\infty])$ descends to an object

$$\big(M_0 = [L_0 \to G_0], \mathcal{B}, \varepsilon_0 : \mathcal{B}_0 \xrightarrow{\sim} M_0[p^\infty]\big)$$

in $\mathrm{Def}(R, k)$. For any $\sigma \in \Gamma$ we then have an isomorphism in $\mathrm{Def}(R', k')$

$$(\varphi_{\sigma,0}, \psi_\sigma) : (M'_0, \mathcal{B}', \mathrm{can}) \xrightarrow{\sim} (\sigma^* M'_0, \sigma^* \mathcal{B}', \sigma^* \mathrm{can}),$$

where

$$\varphi_{\sigma,0} : M'_0 \xrightarrow{\sim} \sigma^* M'_0, \qquad \psi_\sigma : \mathcal{B}' \xrightarrow{\sim} \sigma^* \mathcal{B}',$$

make the diagram

$$\begin{array}{ccc} \mathcal{B}'_0 & \xrightarrow{\psi_{\sigma,0}} & \sigma^* \mathcal{B}'_0 \\ {\scriptstyle \mathrm{can}} \downarrow & & \downarrow {\scriptstyle \sigma^* \mathrm{can}} \\ M'_0[p^\infty] & \xrightarrow{\varphi_{\sigma,0}[p^\infty]} & \sigma^* M'_0[p^\infty] \end{array}$$

commute. By the full faithfulness of (1.1) proved in Corollary 4.2 and Proposition 4.3, the isomorphism $(\varphi_{\sigma,0}, \psi_\sigma)$ gives a unique isomorphism $\varphi_\sigma : M' \xrightarrow{\sim} \sigma^* M'$ that lifts $\varphi_{\sigma,0}$ and restricts to ψ_σ on BT groups. Hence, we have defined an action of Γ on M' compatible with the Γ-action on R' and thus M' descends over R, as explained in Section 3.1. ■

Proposition 4.5 *The functor (1.1) is essentially surjective.*

Proof Let $(M_0 = [L_0 \to G_0], \mathcal{B}, \varepsilon_0)$ be an object of $\mathrm{Def}(R, k)$.

As a first step, consider the case where $L_0 = 0$. Thanks to Lemma 4.4 we can assume that the maximal subtorus T_0 of G_0 is split of dimension d. The Cartier dual of G_0 is a 1-motive $[w_0 : \mathbb{Z}^d \to A_0^*]$, where A_0^* is the dual of the abelian quotient A_0 of G_0. The abelian variety A_0 lifts to an abelian scheme A over R, and since the reduction map $A(R) \to A(k)$ is surjective, the morphism w_0 lifts to a morphism $w : \mathbb{Z}^d \to A^*$. Passing to Cartier duals, we obtain an R-group scheme G that is an extension of A by $\mathbb{G}_{m,R}^d$ and lifts G_0. Now, the BT group $G[p^\infty]$ might not be isomorphic to $\mathcal{B}$. Repeating word by word the proof of the classical Serre-Tate theorem [6, Thm. 1.2.1, pp. 145–146], one finds a finite flat subgroup scheme K of $G[p^{2s}]$ such that G/K is a lifting of G_0 with BT group isomorphic to $\mathcal{B}$.

The case when $M_0 = M_{0,\mathrm{ab}}$ follows immediately by Cartier duality from the previous step.

For the general case, we can assume that L_0 is constant and the maximal torus in G_0 is split, again by Lemma 4.4. Recall that by Lemma 2.2 the BT group $\mathcal{B}$ is naturally filtered so that $W_{-1}\mathcal{B}$ is a lifting of $G_0[p^\infty]$ and $\mathcal{B}/W_{-2}\mathcal{B}$ is a lifting of $M_{0,\mathrm{ab}}[p^\infty]$. By the previous steps we know that G_0 lifts to an R-scheme G, which is an extension of an abelian scheme A by $\mathbb{G}_{m,R}^d$, and $G[p^\infty] = W_{-1}\mathcal{B}$; further $M_{0,\mathrm{ab}}$ lifts to a 1-motive $M_A = [u_A : \mathbb{Z}^m \to A]$ whose BT group is isomorphic to $\mathcal{B}/W_{-2}\mathcal{B}$.

Let $M' = [u': \mathbb{Z}^m \to G]$ be any extension of M_A by T; it exists, since $H^1(R, \mathbb{G}_{m,R}) = 0$. Since $T(R) \to T(k)$ is surjective, we can assume that u' is also a lifting of u_0. We are then left to alter u' so that $M'[p^\infty] \simeq \mathcal{B}[p^\infty]$ in $\mathrm{Ext}_R((\mathbb{Q}_p/\mathbb{Z}_p)^m, G[p^\infty])$.

Let $\mathcal{E} = \mathcal{B}[p^\infty] - M'[p^\infty]$, as extension of $(\mathbb{Q}_p/\mathbb{Z}_p)^m$ by $G[p^\infty]$. Since the push-out along $G[p^\infty] \to A[p^\infty]$ maps $\mathcal{E}$ to the trivial extension of $(\mathbb{Q}_p/\mathbb{Z}_p)^m$ by $A[p^\infty]$, there exists by Lemma 2.4 a 1-motive $N = [v: \mathbb{Z}^m \to T]$ such that $v_0 = 0$ and the push-out of $N[p^\infty]$ along $T[p^\infty] \to G[p^\infty]$ is $\mathcal{E}$. Then $M = [u = u' + v: \mathbb{Z}^m \to G]$ is a lifting of M_0 and $M[p^\infty]$ is isomorphic to $\mathcal{B}[p^\infty]$. ∎

With this proposition, the proof of Theorem 1.1 is completed.

Remark 4.6 Note that some results on deformations of 1-motives (but not Theorem 1.1) were proved with other methods in Madapusi's thesis [7]; however, they are not included in the preprint written by the author under the name K. Madapusi Pera and bearing the same title of the thesis.

Since any extension of an étale BT group by a toroidal BT group is split over k, we deduce from Theorem 1.1 the following generalization of Lemma 2.4.

Corollary 4.7 *Suppose T is an R-torus and L is a lattice. Given a 1-motive $M_0 = [u_0: L_0 \to T_0]$ and any BT group $\mathcal{B}$ that is an extension of $L[p^\infty]$ by $T[p^\infty]$ there is a unique 1-motive $M = [u: L \to T]$ that lifts M_0 and whose BT group is isomorphic to $\mathcal{B}$.*

4.3 Formal Moduli and Serre–Tate Coordinates

Let k be an algebraically closed field of characteristic $p > 0$. We say that a 1-motive M_0 over k is ordinary if $\mathrm{gr}_{-1} M_0 = A_0$ is an ordinary abelian variety (possibly trivial). Thanks to Theorem 1.1 one can easily extend Serre–Tate theory for ordinary abelian varieties to ordinary 1-motives. In this section, we will illustrate some results in this direction. Proofs are only sketched, since no new phenomena appear.

Following [6, § 2] one can define the formal moduli of a given 1-motive M_0 over k. Namely, let $\widehat{\mathscr{M}}_{M_0} = \widehat{\mathscr{M}}$ be the functor

$$\widehat{\mathscr{M}}(R) := \{R\text{-liftings of } M_0\}/\mathrm{iso},$$

where R is an artinian local ring with residue field k. By Theorem 1.1 we get a bijection

$$\widehat{\mathscr{M}}(R) = \{R\text{-liftings of } M_0[p^\infty]\}/\mathrm{iso}.$$

Theorem 4.8 *Let M_0, N_0 be ordinary 1-motives over k.*

(i) *Let R be an artinian local ring as above. For any 1-motive M over R lifting M_0, there exists a canonical $\mathbb{Z}_p$-bilinear form (the Serre–Tate coordinates)*

$$q(M/R, \cdot, \cdot): T_p M_0(k) \otimes T_p M_0^*(k) \longrightarrow \widehat{\mathbb{G}}_{\mathrm{m}}(R),$$

where M_0^* denotes the Cartier dual of M_0 and $T_p M_0(k) := \varprojlim_n M_0[p^n](k)$. It induces an isomorphism of functors

$$\widehat{\mathscr{M}}(\cdot) \simeq \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p M_0(k) \otimes T_p M_0^*(k), \widehat{\mathbb{G}}_{\mathrm{m}}(\cdot)\right).$$

(ii) *Let $\varphi_0\colon M_0 \to N_0$ be a morphism of 1-motives and let M, N be liftings over R of M_0, N_0, respectively. Then φ_0 lifts to an R-morphism $\varphi\colon M \to N$ if and only if*

$$q(M/R, \alpha, \varphi_0^*(\beta)) = q(N/R, \varphi_0(\alpha), \beta)$$

for every $\alpha \in T_pM_0(k)$ and $\beta \in T_pM_0^*(k)$. Further, if a lifting exists, it is unique.

Proof The proof goes exactly as in the classical case. For the convenience of the reader we point out the main steps (*cf.* [6, p. 152]).

- Since k is algebraically closed and M_0 is ordinary, the BT groups $M_0[p^\infty]^\circ$ is split multiplicative and $M_0[p^\infty]^{\text{ét}}$ is split étale. Hence, the same are true for $M[p^\infty]^\circ$ and $M[p^\infty]^{\text{ét}}$ for any lifting M of M_0 over R. For this reason we may write $M_0[p^\infty]^\circ$ (resp. $M_0[p^\infty]^{\text{ét}}$) for the corresponding functor on the category of artinian local rings with residue field k.
- By Cartier duality, there is a perfect pairing $M_0[p^n] \times M_0^*[p^n] \to \mu_{p^n}$ [4, 10.2.5], inducing an isomorphism of functors

$$M_0[p^\infty]^\circ \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}_p}(T_pM_0^*(k), \widehat{\mathbb{G}}_m),$$

  on the category of artinian local rings with residue field k. We denote the corresponding pairing by $E_M\colon M_0[p^\infty]^\circ \times T_pM_0^*(k) \to \widehat{\mathbb{G}}_m$.
- By [8, Proposition 2.5 p. 180] and the previous steps, for any R as above and any lifting M of M_0 over R, we have isomorphisms

$$\begin{aligned}\operatorname{Ext}_R(M[p^\infty]^{\text{ét}}, M[p^\infty]^\circ) &= \operatorname{Ext}_R(T_pM_0(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p, M[p^\infty]^\circ)\\ &\simeq \operatorname{Hom}\big(T_pM_0(k), M[p^\infty]^\circ(R)\big);\end{aligned}$$

  thus, there exists a unique $\phi_M \in \operatorname{Hom}_R(T_pM_0(k), M[p^\infty]^\circ(R))$ associated with the class of the BT group $M[p^\infty]$, viewed as extension of $M[p^\infty]^{\text{ét}}$ by $M[p^\infty]^\circ$.

Then we can define $q(M/R, \alpha, \beta) := E_M(\phi_M(\alpha), \beta)$ in $\widehat{\mathbb{G}}_m(R)$ and follow word by word the proof in [6]. ■

Remark 4.9 Let M_0 be an ordinary 1-motive over K and denote its graded quotients (by the weight filtration) by L_0, A_0 and T_0. Fix basis: $\alpha_1, \dots, \alpha_g$ of $T_pA_0(k)$ with $g = \dim A_0$; $\alpha_{g+1}, \dots, \alpha_{g+\ell}$ of $T_pL_0(k)$ with $\ell = \operatorname{rank} L_0$; $\beta_1, \dots, \beta_g$ of $T_pA_0^*(k)$; $\beta_{g+1}, \dots, \beta_{g+t}$ of $T_pT_0^*(k)$ with $t = \dim T_0$. Then the coordinate ring of $\widehat{\mathscr{M}}$ is isomorphic to

$$W(k)[[T_{i,j}]] \quad \text{where } T_{i,j} = q(\alpha_i, \beta_j) - 1,$$

where $q(\cdot, \cdot)$ is the bilinear form associated with the universal formal deformation. In particular $\widehat{\mathscr{M}}$ is represented by a $(g+\ell) \times (g+t)$-dimensional formal torus.

Proposition 4.10 *Let M_0 be an ordinary 1-motive over k. Then there exists a 1-motive M over $W(k)$ lifting M_0 such that $\operatorname{End}_{\mathcal{M}_1(W(k))}(M) \to \operatorname{End}_{\mathcal{M}_1(k)}(M_0)$ is bijective.*

Proof By Theorem 1.1 there exists a unique 1-motive $M_n = [L_n \to G_n]$ over $W_{n+1}(k)$ lifting M_0 such that $M_n[p^\infty]$ has split connected-étale sequence. As n goes to infinity, these liftings form a compatible system. In order to see that the limit is algebraizable it is enough to check that $\varinjlim_n G_n$ is algebraizable, since all L_n are torsion-free constant abelian groups of the same rank, and for any group scheme J of finite type over $W(k)$

$$J(W(k)) = \varprojlim_m J\big(W_m(k)\big). \tag{4.1}$$

Now, by Cartier duality, the system $(G_n)_n$ corresponds to a compatible system of 1-motives $[T_n^* \to A_n^*]$. The latter is algebraizable if the system $(A_n^*)_n$ is. Since $M_n[p^\infty]$ has split connected-étale sequence, the same are true for $A_n[p^\infty]$ and $A_n^*[p^\infty]$. So $(A_n^*)_n$ is algebraizable by the proof of [8, Ch. 5, Thm 3.3, p. 173].

Let us denote by $M = [u: L \to G]$ (resp. A) the lift of M_0 (resp A_0) over $W(k)$ constructed in the previous paragraph; then G is extension of A by the unique torus T lifting T_0. By construction and Theorem 1.1, any reduction map $\mathrm{End}_{\mathcal{M}_1(W_{n+1}(k))}(M_n) \to \mathrm{End}_{\mathcal{M}_1(k)}(M_0)$ is bijective. We are left to check that the map $\mathrm{End}_{\mathcal{M}_1(W(k))}(M) \to \varprojlim \mathrm{End}_{\mathcal{M}_1(W_{n+1}(k))}(M_n) = \mathrm{End}_{\mathcal{M}_1(k)}(M_0)$ is bijective. Let $\varphi = (f, g)$ be an endomorphism of M, *i.e.,* we have morphisms $f: L \to L$, $g: G \to G$ such that $u \circ f = g \circ u$. If $\varphi_0 = 0$, then $\varphi = 0$ by devissage, since

$$\mathrm{End}_{W(k)}(L) = \mathrm{End}_k(L_0), \qquad \mathrm{End}_{W(k)}(T) = \mathrm{End}_k(T_0), \tag{4.2}$$

and $\mathrm{End}_{W(k)}(A) = \mathrm{End}_k(A_0)$ by [8, Ch. 5, Thm 3.3]. For surjectivity, let $\varphi_n = (f_n, g_n): M_n \to M_n$ be a compatible system of endomorphisms. By (4.2) and (4.1) it suffices to show the existence of $g: G \to G$ lifting the morphisms $(g_n)_n$. We can work with the Cartier dual $[T^* \to A^*]$ of G instead, and lift the Cartier duals of $(g_n)_n$; as above, we can reduce to the case $T^* = 0$ and then conclude by [8, Ch. 5, Thm 3.3]. ■

Acknowledgements This work was done while the second author was visiting the University of Padua supported by a "délégation CNRS". The first author thanks the project PRIN 2015 "Number Theory and Arithmetic Geometry" for financial support. Both authors thank the referee, whose detailed comments and helpful suggestions improved the exposition of this paper.

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