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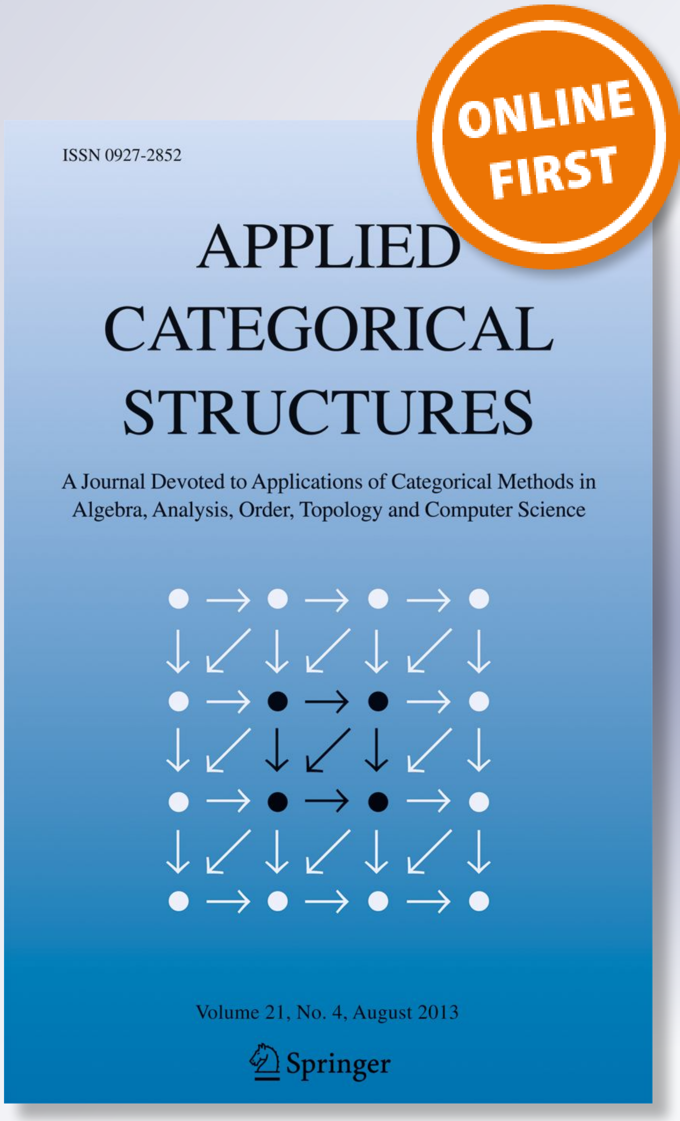

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Extensions of Filtered Ogus Structures

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Abstract
We compute the Ext group of the (filtered) Ogus category over a number field K. In particular we prove that the filtered Ogus realisation of mixed motives is not fully faithful.

Keywords Filtered Ogus structures · Extensions · de Rham

Mathematics Subject Classification 13D09 · 14F42 · 14F30

1 Introduction

Recently Andreatta, Barbieri-Viale and Bertapelle [1] have defined the filtered Ogus realisation $T_{\mathbf{FOg}}$ for 1-motives over a number field K. In fact by [5] there exists a cohomology theory for K-varieties with values in $\mathbf{FOg}(K)$ compatible with $T_{\mathbf{FOg}}$. More precisely let $\mathbf{DM}_{gm}(K)$ be the Voevodsky's category of geometric motives over K, then there exists a (homological) realisation functor

$$R_{\mathbf{FOg}} : \mathbf{DM}_{gm}(K) \to \mathrm{D}^b(\mathbf{FOg}(K))$$

compatible with $T_{\mathbf{FOg}}$.

The aim of this paper is to compute the Ext group in **FOg** (Proposition 3.2). We follow the method of Beilinson [2,3].

It follows (see Remark 3.6) that the filtered Ogus realisation of mixed motives is not fully faithful in general, even though $T_{\mathbf{FOg}}$ is fully faithful.

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1.1 Notations and Conventions

Throughout this article, K will denote a number field. A place of K will always mean a finite place (we will never need to consider real or complex places). For every such place v of K, let K_v denote the completion, $\mathcal{O}_v$ the ring of integers, k_v the residue field, p_v its characteristic, and $q_v = p_v^{n_v}$ its order. For all v which are unramified over $\mathbb{Q}$, let σ_v denote the lift to K_v of the absolute Frobenius of k_v.

2 The Categories

2.1 The Ogus Category

Let P be a cofinite set of absolutely unramified places of K. We define $\mathcal{C}_P$ to be the category whose objects are systems $M = (M_{\mathrm{dR}}, (M_v, \phi_v, \epsilon_v)_{v \in P})$ such that:

(1) M_{dR} is a finite dimensional K-vector space;
(2) (M_v, ϕ_v) is a F-K_v-isocrystal, that is, M_v is equipped with a σ_v-linear automorphism ϕ_v;
(3) $\epsilon = (\epsilon_v)_{v \in P}$ is a system of K_v-linear isomorphisms

$$\epsilon_v : M_{\mathrm{dR}} \otimes K_v \to M_v \ .$$

A morphism $f : M \to M'$ is then a collection $(f_{\mathrm{dR}}, (f_v)_{v \in P})$ where:

(1) $f_{\mathrm{dR}} : M_{\mathrm{dR}} \to M'_{\mathrm{dR}}$ is a K-linear map;
(2) $f_v : M_v \to M'_v$ is K_v-linear morphism compatible with Frobenius and such that $\epsilon_v^{-1} \circ f_v \circ \epsilon_v = f_{\mathrm{dR}} \otimes K_v$.

Note that by the second criterion, to specify a morphism it is enough to specify f_{dR}. There are obvious 'forgetful' functors $\mathcal{C}_P \to \mathcal{C}_{P'}$ whenever $P' \subset P$ and we can form the Ogus category $\mathbf{Og}(K)$ as the 2-colimit

$$\mathbf{Og}(K) = 2 \operatorname*{colim}_P \mathcal{C}_P$$

where P varies over all cofinite sets of unramified places of K. For an object $M \in \mathbf{Og}(K)$ and $n \in \mathbb{Z}$ we denote by $M(n)$ the Tate twist of M, that is where each Frobenius ϕ_v is multiplied by p_v^{-n}.

2.2 Weights

A *weight filtration* on an object $M = (M_{\mathrm{dR}}, (M_v, \phi_v, \epsilon_v)_{v \in P}) \in \mathcal{C}_P$ is an increasing filtration $W_\bullet M$ by sub-objects in $\mathcal{C}_P$ such that for all $v \in P$ the graded pieces $\mathrm{Gr}_i^W M_v$ are pure of weight i. That is, all eigenvalues of the linear map $\phi_v^{n_v}$ are Weil numbers of q_v-weight i (i.e. all their conjugates have absolute value $q_v^{i/2}$ [4]). Again, to give a weight filtration on M it suffices to give a filtration on M_{dR} which induces a weight filtration on all M_v.

2.3 The Filtered Ogus Category

We can therefore consider the filtered Ogus category $\mathbf{FOg}(K)$ whose objects are objects of $\mathbf{Og}(K)$ equipped with a weight filtration, and morphisms are required to be compatible with this filtration.

Lemma 2.1 ([1], Lemma 1.3.2) *The filtered Ogus category* $\mathbf{FOg}(K)$ *is a* $\mathbb{Q}$*-linear abelian category, and the forgetful functor*

$$\mathbf{FOg}(K) \to \mathbf{Og}(K)$$

is fully faithful.

2.4 Internal Hom

If M, N are two objects in $\mathbf{FOg}$ then we can define the internal Hom, denoted by $\underline{\mathrm{Hom}}_{\mathbf{FOg}}(M, N)$ as follows:

(1) $\underline{\mathrm{Hom}}_{\mathbf{FOg}}(M, N)_{\mathrm{dR}} := \underline{\mathrm{Hom}}_K(M_{\mathrm{dR}}, N_{\mathrm{dR}})$ is just the usual Hom of K-vector spaces.
(2) for all v, $\underline{\mathrm{Hom}}_{\mathbf{FOg}}(M, N)_v := \underline{\mathrm{Hom}}_{K_v}(M_v, N_v)$ and for almost all v this K_v-vector space is endowed with the Frobenius $f \mapsto \phi_v^N \circ f \circ (\phi_v^M)^{-1}$
(3) $W_r \underline{\mathrm{Hom}}_{\mathbf{FOg}}(M, N) := \{f \in \underline{\mathrm{Hom}}_{\mathbf{FOg}}(M, N) : f(W_i M) \subset W_{i+r} N\}$.

3 Ext Computation

Let M, N be two objects in $C^b(\mathbf{FOg})$ (the category of bounded complexes of $\mathbf{FOg}$) and consider the following complexes

$$\begin{aligned} A(M, N) &= W_0 \underline{\mathrm{Hom}}^\bullet(M, N)_{\mathrm{dR}} \\ &= W_0 \underline{\mathrm{Hom}}_K^\bullet(M_{\mathrm{dR}}, N_{\mathrm{dR}}) \\ B(M, N) &= \prod_v{}' W_0 \underline{\mathrm{Hom}}^\bullet(M, N)_v \quad \text{(restricted product)} \end{aligned}$$

and the morphism

$$\xi_{M,N} : A(M, N) \to B(M, N) \qquad \xi(x) = (x\phi_M - \phi_N x), \ .$$

We want to prove that the cone of this map compute the ext-groups of $\mathbf{FOg}$, i.e.

$$\mathrm{Ext}^i_{\mathbf{FOg}}(M, N) \cong H^{i-1}(\mathrm{Cone}(\xi_{M,N})) \ .$$

Lemma 3.1 *Let* $\xi_{M,N}$ *as above, then for any* i *and for any element* $b \in B^i(M, N)$ *there exist a quasi-isomorphism* $N \to E$ *of complexes such that the image of* b *in* $\mathrm{Coker}(\xi_{M,E})$ *is zero.*

Proof Take $b \in B^0(M, N)$, so that $b = (b^i)$ with $b^i \in \prod_v' W_0 \underline{\mathrm{Hom}}(M^i, N^i)_v$. Then we construct E as follows

$$E := \mathrm{Cone}((0, \mathrm{id}) : M[-1] \to N \oplus M[-1])$$

where everything is defined as expected but the Frobenius: ϕ_E on $E^i = N^i \oplus M^{i-1} \oplus M^i$ is given by

$$\begin{aligned} \phi_E(x, 0, 0) &= (\phi_N(x), 0, 0) \\ \phi_E(0, y, 0) &= (b^i d_M y - d_N b^{i-1}, \phi_M(y), 0) \\ \phi_E(0, 0, z) &= (-b^i z, 0, \phi_M z) \end{aligned}$$

By construction $N \to E$ is a quasi-isomorphism and there is a short exact sequence

$$0 \to N \to E \to \mathrm{Cone}(\mathrm{id}_M)[-1] \to 0\,.$$

Finally we remark that the natural map $B^0(M,N) \to B^0(M,E)$ sends b to $(b,0,0)$ and we can explicitly compute

$$\xi_{M,E}(0,0,\mathrm{id}) = (0,0,\mathrm{id})\phi_M - \phi_E(0,0,\mathrm{id}) = (0,0,\phi_M) - (-b,0,\phi_M) = (b,0,0)$$

as expected. □

Proposition 3.2 *Let M, N be two complexes in $C^b(\mathbf{FOg})$*

$$\mathrm{Ext}^i_{\mathbf{FOg}}(M,N) \cong H^{i-1}(\mathrm{Cone}(\xi_{M,N}))\,.$$

Proof The proof is similar to [2, Proposition 1.7]. We have by definition

$$\mathrm{Ext}^i_{\mathbf{FOg}}(M,N) = \mathrm{Hom}_{D^b(\mathbf{FOg})(M,N[i])} = \operatorname*{colim}_I \mathrm{Hom}_{K^b(\mathbf{FOg})}(M,L[i])$$

where I is the category of quasi-isomorphisms $s : N \to L$ in the homotopy category $K^b(\mathbf{FOg})$.

By the octahedron axiom and the exactness of $A(M,-)$, $B(M,-)$ there is a long exact sequence

$$H^i(\ker \xi_{M,N}) \to H^i(\mathrm{Cone}(\xi_{M,N})[-1]) \to H^i(\mathrm{coker}(\xi_{M,N})[-1]) \to +\,.$$

Note that $H^i(\ker \xi_{M,N}) = \mathrm{Hom}_{K^b(\mathbf{FOg})}(M,N[i])$. By the previous lemma

$$\operatorname*{colim}_I H^i(\mathrm{coker}(\xi_{M,L})[-1]) = 0\,.$$

Thus we obtain the expected result by taking the colimit over I of the above long exact sequence.

We can also give a direct proof in the case of chain complexes concentrated in degree zero, as explained in the following remark. □

Remark 3.3 When $M, N \in \mathbf{FOg}$ we can derive the above formula as follows. Let

$$0 \to N \to E \xrightarrow{\pi} M \to 0$$

be an extension in $\mathbf{FOg}$. Choose a section $s_{\mathrm{dR}} \in W_0\,\underline{\mathrm{Hom}}(M,E)_{\mathrm{dR}}$ of $\pi_d R$. After base change to K_v we get sections $s_v \in W_0\,\underline{\mathrm{Hom}}(M,E)_v$ and we can define (for almost all v)

$$x_v := s_v \circ \phi_{M_v} - \phi_{E_v} \circ s_v\,.$$

It follows that $x_v \in W_0\,\underline{\mathrm{Hom}}(M,N)_v$ so that $x = (x_v)_v$ is an element of $\prod'_v W_0\,\underline{\mathrm{Hom}}(M,N)_v$. Starting with another section s'_{dR} we will get another x' and the difference $x - x'$ lies in $(\circ\phi_M - \phi_N\circ)W_0\,\underline{\mathrm{Hom}}(M,N)_{\mathrm{dR}}$ by construction. Then we easily get a map

$$\Phi : \mathrm{Ext}^1_{\mathbf{FOg}}(M,N) \to \frac{\prod'_v W_0\,\underline{\mathrm{Hom}}(M,N)_v}{(\circ\phi_M - \phi_N\circ)W_0\,\underline{\mathrm{Hom}}(M,N)_{\mathrm{dR}}}, \quad \Phi(E) = (x_v)_v$$

Moreover given a family $x = (x_v)_v$ as above we can define the extension E_x to be the direct sum $N \oplus M$ except for the fact that we set the Frobenius to be

$$\phi_{E,v}(n,m) := (\phi_{N,v}(n) - x_v(m), \phi_{M,v}(m))\,.$$

By construction we have $\Phi(E_x) = x$ and we prove that Φ is an isomorphism.

Proposition 3.4 *Let $M, N \in \mathbf{FOg}$ there is a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbf{FOg}}(M, N) \to \mathrm{Ext}^1_{\mathbf{Og}}(M, N) \to \frac{\prod'_v W_{\geq 1}\, \underline{\mathrm{Hom}}(M, N)_v}{(\circ\phi_M - \phi_N \circ)\, \underline{\mathrm{Hom}}(M, N)_{\mathrm{dR}}} \to 0\,.$$

Proof The methods we have introduced to compute the extension groups in $\mathbf{FOg}_K$ work also for $\mathbf{Og}_K$. In fact we consider the above construction forgetting about weights

$$A'(M, N) = \underline{\mathrm{Hom}}^\bullet(M, N)_{\mathrm{dR}}$$

$$B'(M, N) = \prod_v{}' \underline{\mathrm{Hom}}^\bullet(M, N)_v$$

$$\xi'_{M,N} : A'(M, N) \to B'(M, N) \qquad \xi(x) = (x\phi_M - \phi_N x)$$

so that we have

$$\mathrm{Ext}^i_{\mathbf{Og}}(M, N) \cong H^{i-1}(\mathrm{Cone}(\xi'_{M,N}))\,.$$

In particular if $M, N \in \mathbf{FOg}$ there is an exact sequence of complexes

$$0 \to \mathrm{Cone}(\xi_{M,N}) = W_0\,\mathrm{Cone}(\xi'_{M,N}) \to \mathrm{Cone}(\xi'_{M,N}) \to W_{\geq 1}\,\mathrm{Cone}(\xi'_{M,N}) \to 0$$

whose associated long exact sequence degenerates to the short exact sequence of the statement. □

Remark 3.5 Let us consider an intermediate category $\mathbf{FOg} \subset \mathbf{FOg}' \subset \mathbf{Og}$ whose objects are $M \in \mathbf{Og}(K)$ endowed with an increasing filtration $M_i \subset M_{i+1}$ (without any condition on Frobenius eigenvalues). This is just an exact category and it is not full in $\mathbf{Og}$. Nevertheless $\mathbf{FOg} \subset \mathbf{Og}$ is full and for two objects $M, N \in \mathbf{FOg}$ we have

$$\mathrm{Ext}^1_{\mathbf{FOg}}(M, N) \cong \mathrm{Ext}^1_{\mathbf{FOg}'}(M, N)$$

just following the previous proof.

Remark 3.6 It follows from the previous proposition that the $\mathrm{Ext}^1_{\mathbf{FOg}}(M, N)$ are not countable in general and in particular different from motivic cohomology. For instance already for $K = \mathbb{Q}$ we get

$$\mathrm{Ext}^1_{\mathbf{FOg}}(\mathbb{Q}, \mathbb{Q}(1)) \cong \frac{\{(a_p)_p \in \prod'_p \mathbb{Q}_p\}}{\{(b - p^{-1}b)_p : b \in \mathbb{Q}\}}\,,$$

which is uncountable and so different from $\mathrm{Ext}^1_{\mathbf{DM}}(\mathbb{Q}, \mathbb{Q}(1)) = \mathbb{Q}^* \otimes \mathbb{Q}$. Hence the filtered Ogus realisation of mixed motives in not full.

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