On log differentials of local fields

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Abstract – We prove a logarithmic version of Fontaine's classic result on differentials of $\mathcal{O}_{\overline{K}}$ over $\mathcal{O}_K$.

1. Introduction

Let K be a p-adic field, $\overline{K}$ an algebraic closure and $\mathbb{C}_p = \widehat{\overline{K}}$ its completion. We denote by $\mathcal{O}_K$ and $\mathcal{O}_{\overline{K}}$ their rings of integers. The module of Kähler differentials $\Omega = \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is well-known. In particular Fontaine [3, Corollaire 1] gives the following identification

$$\mathbb{C}_p(1) \cong V_p(\Omega)$$

where $V_p(\Omega)$ is the rational Tate module of Ω. The latter isomorphism is induced by the map

$$\mathrm{dlog}_F\colon \overline{K}(1) = \overline{K} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \longrightarrow \Omega, \quad \mathrm{dlog}_F(p^{-n} \otimes \underline{\epsilon}) = \frac{d\epsilon_n}{\epsilon_n},$$

where $\underline{\epsilon} = (\epsilon_n)_{n \geq 0}$ is a compatible sequence of p-powers roots of 1, with $\epsilon_1 \neq 1$.

In this article we slightly modify the source of the previous map in order to compute a logarithmic version of Ω and its Tate module.

Fix $q \in p\mathcal{O}_K \setminus \{0\}$. We denote by E_q the Tate elliptic curve of parameter q: its group of $\overline{K}$-points is isomorphic to $\overline{K}^*/\langle q \rangle$ and its Tate module, denoted $\mathbb{Z}_p(1)^{\mathrm{log}}$,

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is an extension of the trivial Galois module $\mathbb{Z}_p$ by $\mathbb{Z}_p(1)$. Its divisible group is denoted by $\mu_{p^\infty}^{\log}$. Fix a sequence $\underline{q} = (q_n)_n$ of compatible p-power roots of q. Then, $\mathbb{Z}_p(1)^{\log} = \mathbb{Z}_p\tilde{\epsilon} \oplus \mathbb{Z}_p\tilde{q}$ as a $\mathbb{Z}_p$-module, where $\tilde{\epsilon}, \tilde{q}$ are the logarithms of $\underline{\epsilon}, \underline{q}$. The Galois action is given by $\begin{pmatrix} \chi & c \\ 0 & 1 \end{pmatrix}$, where $\sigma(q_n) = \epsilon_n^{c_n(\sigma)} q_n$ (cf. [1, § II.4]).

The inclusion $\{q_n^s : s, n \in \mathbb{N}\} \subset \mathcal{O}_{\bar{K}}$ gives a pre-log-structure $\bar{N}$ on $\mathcal{O}_{\bar{K}}$, and we define $\Omega^{\log} = \Omega_{(\mathcal{O}_{\bar{K}}, \bar{N})/(\mathcal{O}_K, \text{triv})}$ (see § 2.1).

For any $\mathbb{Z}_p$-module M we denote $M(1)^{\log} = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\log}$ (and similarly without $^{\log}$).

We aim to prove the following result (cf. [3, Théorème 1']).

THEOREM 1.1. *There is a natural surjection*

$$\text{dlog} \colon \bar{K}(1)^{\log} \longrightarrow \Omega^{\log}$$

compatible with dlog_F. *Moreover it induces an isomorphism*

$$\mathcal{O}_{\mathbb{C}_p}(1)^{\log} \cong T_p(\Omega^{\log}).$$

In particular $V_p(\Omega^{\log})$ can be seen as a submodule of $\mathbb{B}_{\text{st}}$ sitting in the following exact sequence

$$0 \longrightarrow \mathbb{C}_p \cdot t \longrightarrow V_p(\Omega^{\log}) \longrightarrow \mathbb{C}_p \cdot u \longrightarrow 0,$$

where $t = \log([\epsilon^\flat])$ and $u = \log([\underline{q}^\flat])$ (we write $\underline{a}^\flat$ to denote an element of $\mathbb{C}_p^\flat$ given by a fixed compatible system $\underline{a}$ of p-powers roots of a).

The result of Fontaine is used to compute the p-adic periods of abelian varieties with good reduction. We believe that the logarithmic version can be used to address the case of semistable abelian varieties in a direct way (see remark 2.1).

2. Logarithmic differentials

We recall some definitions from [4]. A logarithmic ring (A, M, α) is the data of: a ring A; a monoid M; a morphism $\alpha \colon M \to A$ of monoids[1] inducing an isomorphism $\alpha^{-1}(A^\times) \cong A^\times$. If M is a monoid, we denote by M^+ its group completion. Given a morphism of logarithmic rings $(A, M, \alpha) \to (B, N, \beta)$ we define the module of log differentials $\Omega_{(B,N)/(A,N)}$ to be the quotient of the module

$$\Omega_{B/A} \oplus (B \otimes \text{coker}(M^+ \longrightarrow N^+))$$

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[1] With respect to the multiplication on A.

by the submodule generated by elements of the form $(d\beta(n), 0) - (0, \beta(n) \otimes n)$ for $n \in N$. There are natural maps $d \colon B \to \Omega_{(B,N)/(A,N)}$ (the usual differential) and $\mathrm{dlog}_N \colon N \to \Omega_{(B,N)/(A,N)}$ such that $\beta(n) \, \mathrm{dlog}_N(n) = d\beta(n)$, for all $n \in N$.

2.1 – *The log structure on $\mathcal{O}_{\bar{K}}$*

Fix $q \in p\mathcal{O}_K$, we also fix a system $\underline{q} = (q_n)_n$ of compatible p-power roots of q, i.e. $q_0 = q$ and $q_{n+1}^p = q_n$ for all n. We define the following monoid

$$\bar{N} := \underline{q}^{\mathbb{N}[1/p]}, \quad \text{where } \underline{q}^{m/p^s} = q_s^m.$$

Then we have the pre-log-structure $\bar{N} \to \mathcal{O}_{\bar{K}}$. On $\mathcal{O}_K$ we may consider pre-log-structure $N = 1$.

2.2 – *The map* dlog

By definition of $\Omega^{\log}$ there is a natural map $\mathrm{dlog} \colon \mu_{p^\infty}^{\log} \to \Omega^{\log}$ sending $\epsilon_r^i \cdot q_s^j$ to the image of

$$\left(\frac{d\epsilon_r^i}{\epsilon_r^i}, 1 \otimes q_s^j \right).$$

Clearly we have $\mathrm{dlog}(\epsilon_r \cdot q_s) = \mathrm{dlog}_F(\epsilon_r) + \mathrm{dlog}_{\bar{N}}(q_s)$.

We denote by the same symbol the $\mathcal{O}_{\bar{K}}$-linearisation. Notice that we can identify $\mathcal{O}_{\bar{K}} \otimes_{\mathbb{Z}_p} \mu_{p^\infty}^{\log} \cong (\bar{K}/\mathcal{O}_{\bar{K}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\log}$ so the previous map can be naturally lifted to

$$\mathrm{dlog} \colon \bar{K}(1)^{\log} \longrightarrow \Omega^{\log}.$$

The previous map is compatible with the dlog_F defined by Fontaine: namely there is a morphism of exact sequences

$$
\begin{array}{ccccccccc}
0 & \to & \mathfrak{a}(1) & \longrightarrow & \bar{K}(1) & \xrightarrow{\mathrm{dlog}_F} & \Omega & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & \bar{K}(1)^{\log} & \xrightarrow{\mathrm{dlog}} & \Omega^{\log} & \to & 0
\end{array}
$$

where $\mathfrak{a} = \{ a \in \bar{K} \colon v(a) \geq -v(D_{K/K_0}) - (p-1)^{-1} \}$ ([3, Théorème 1 (ii)]) and $Z = \ker(\mathrm{dlog})$.

Now we are ready to complete the proof of the theorem.

Proof of the Theorem. Consider the following commutative diagram

(1)

$$
\begin{array}{ccccc}
\mathfrak{a}\tilde{\epsilon} & \longrightarrow & \bar{K}\tilde{\epsilon} & \longrightarrow & \Omega \\
\downarrow & & \downarrow & & \downarrow \\
Z & \longrightarrow & \bar{K}(1)^{\log} & \longrightarrow & \Omega^{\log} \\
\downarrow & & \downarrow & & \downarrow \\
Z/\mathfrak{a}\tilde{\epsilon} & \longrightarrow & \bar{K}\tilde{q} & \longrightarrow & \Omega^{\log}/\Omega
\end{array}
$$

The first two rows (and the last two columns) of (1) are short exact sequences by construction. Hence by the snake lemma the last row and the first column are short exact sequences too.

We now prove that $(Z/\mathfrak{a}\tilde{\epsilon}) \cong \mathfrak{b}\tilde{q}$, where $\mathfrak{b} = \mathcal{O}_{\bar{K}}$ if $v(q) \geq p$, otherwise

$$
\mathfrak{b} = \left\{ b \in \bar{K} : v(b) \geq \frac{v(q)}{p} - 1 \right\} \quad (v(q) < p).
$$

Let $b = x/p^n$ with $x \in \mathcal{O}_{\bar{K}}$ and $n \geq 1$. Then $\mathrm{dlog}(b \otimes \underline{q})$ is the image of $x \otimes q_n$: it maps to 0 $\Omega^{\log}/\Omega$ if and only if q_n divides x, i.e.

$$
v(x) \geq v(q_n) = \frac{q}{p^n} \iff v(b) \geq v(q_n) = \frac{v(q)}{p^n} - n.
$$

The maximum of $v(q)p^{-n} - n$ is attained at $n = 1$. Since $\mathrm{dlog}(\mathcal{O}_{\bar{K}} \otimes \underline{q})$ is trivial modulo Ω the claim is proved.

Now we can deduce the short exact sequence

$$
0 \longrightarrow (\bar{K}/\mathfrak{a})\tilde{\epsilon} \longrightarrow \Omega^{\log} \longrightarrow (\bar{K}/\hat{\mathfrak{b}})\tilde{q} \longrightarrow 0
$$

inducing a short exact sequence of Tate modules. Since we have an isomorphism of $\mathbb{Z}_p$-modules (for $\mathfrak{c} = \mathfrak{a}, \mathfrak{b}$)

$$
(\bar{K}/\mathfrak{c})[p^n] \cong (p^{-n}\mathfrak{c}/\mathfrak{c}) \cong (\mathfrak{c}/p^n\mathfrak{c}) \cong (\mathcal{O}_{\bar{K}}/p^n)
$$

we can conclude the proof of the theorem. Indeed it is possible to check the Galois action on generators and compare with $\mathcal{O}_{\mathbb{C}_p}(1)^{\log}$. $\qquad\square$

Remark 2.1. It is now easy to define a Galois equivariant pairing

(2) $$ \langle \cdot, \cdot \rangle : T_p E_q \times \mathrm{Fil}^1 H^1_{\mathrm{dR}}(E_q/K) \longrightarrow V_p \Omega^{\log} $$

analogous to the Fontaine paring for abelian varieties with good reduction. We will discuss the details of this construction in a forthcoming work: we will develop the

p-adic Hodge theory for 1-motives, and construct a perfect pairing between the Tate module and the full de Rham cohomology of a 1-motive over K. The theory in the good reduction (or crystalline) case is already explained in [5, § 3.1], we will treat also the (potentially) semi-stable case.

For the sake of the reader we give a sketch of the construction of (2) in the following paragraphs.

Let $M_q = [\mathbb{Z} \xrightarrow{1 \mapsto q} \mathbb{G}_m]$ be the strict 1-motive over K such that $T_p(M_q) = \mathbb{Z}_p(1)^{\log}$ [6, § 4.2] (See also [2, § 1.3.2]). There is a short exact sequence

$$0 \longrightarrow T_p\Omega^{\log} \longrightarrow A_2^{\log} \xrightarrow{\theta} \mathcal{O}_{\mathbb{C}}\langle X^{\pm 1/p^\infty}\rangle \longrightarrow 0,$$

where $A_2^{\log}$ is simply the p-adic completion of $(B_{dR}^+/\mathrm{Fil}^2)[X^{\pm 1/p^\infty}]$; θ is the usual map on B_{dR}^+ and $\theta(X) = 0$.

This induces a short exact sequence (of complexes)

$$0 \longrightarrow \mathrm{Lie}(\mathbb{G}_m) \otimes T_p\Omega^{\log} \longrightarrow M_q(A_2^{\log}) \longrightarrow M_q(\mathcal{O}_{\mathbb{C}}\langle X^{\pm 1/p^\infty}\rangle) \longrightarrow 0.$$

By taking the cones of the multiplication by p^n and the boundary map of the associated long exact sequence we have

$$\rho_n \colon M_q(\mathcal{O}_{\mathbb{C}})[p^n] = (\bar{K}^*/\langle q\rangle)[p^n] \longrightarrow \mathrm{Lie}(\mathbb{G}_m) \otimes (T_p\Omega^{\log}/p^n),$$

since $M_q(\mathcal{O}_{\mathbb{C}}\langle X^{\pm 1/p^\infty}\rangle)[p^n] = M(\mathcal{O}_{\mathbb{C}})[p^n]$. Then taking the limit over n induces a pairing

$$\langle \cdot, \cdot \rangle \colon \mathbb{Z}_p(1)^{\log} \times \mathrm{Fil}^1 H_{dR}^1(\mathbb{G}_m/\mathcal{O}_K) \longrightarrow T_p\Omega^{\log}$$

such that $\langle (x_n)_n, dT/T \rangle = (\mathrm{dlog}(x_n))_n$. To get the claim it is sufficient to invert p and use the isomorphisms given by the rigid uniformisation: $\mathrm{Fil}^1 H_{dR}^1(\mathbb{G}_m/K) \cong \mathrm{Fil}^1 H_{dR}^1(E_q/K)$ and $\mathbb{Z}_p(1)^{\log} \cong T_p E_q$.

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