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Multivariable de Rham representations, Sen theory and p-adic differential equations

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Let K be a complete valued field extension of $\mathbf{Q}_p$ with perfect residue field. We consider p-adic representations of a finite product $G_{K,\Delta} = G_K^\Delta$ of the absolute Galois group G_K of K. This product appears as the fundamental group of a product of diamonds. We develop the corresponding p-adic Hodge theory by constructing analogues of the classical period rings B_{dR} and B_{HT}, and multivariable Sen theory. In particular, we associate to any p-adic representation V of $G_{K,\Delta}$ an integrable p-adic differential system in several variables $\mathsf{D}_{\mathsf{dif}}(V)$. We prove that this system is trivial if and only if the representation V is de Rham. Finally, we relate this differential system to the multivariable overconvergent (φ, Γ)-module of V constructed by Pal and Zábrádi in [20], along classical Berger's construction [5].

1. Introduction

Let K be a finite extension of $\mathbf{Q}_p$, G_K its absolute Galois group, and Δ a finite set. After the work of Scholze and Weinstein [25], [21], the finite product $G_{K,\Delta} := G_K^\Delta$ can be understood as the fundamental group of a diamond: the product $\times_\Delta \mathsf{Spd}\,\mathbf{Q}_p = \mathsf{Spd}\,\mathbf{Q}_p \times \cdots \times \mathsf{Spd}\,\mathbf{Q}_p$ (this diamond is *not* associated to a perfectoid space). It is then natural to consider p-adic representations for this fundamental group, viewed as coefficients for the diamond $\times_\Delta \mathsf{Spd}\,\mathbf{Q}_p$. Hereafter, we work in a slightly more general context: we assume that K is a complete discretely valued extension of $\mathbf{Q}_p$ with perfect residue field.

Of course the study of such representations can be considered in the *classical* framework of p-adic Hodge theory as developed after the work of Fontaine (*cf* [16], [22], [8]), *i.e.* in terms of period rings and (φ, Γ)-modules. This second approach has been pursued in recent works by Zábrádi, Pal, Kedlaya and Carter ([26], [27], [20] and [12]) in terms of multivariable (multivariate) (φ, Γ)-modules associated to p-adic representations of $G_{K,\Delta}$.

In this article, we develop a multivariable Sen theory in this framework, and construct multivariable (multivariate) p-adic period rings $\mathsf{B}_{\mathrm{dR},\Delta}$ and $\mathsf{B}_{\mathrm{HT},\Delta}$. To any p-adic representation V of $G_{K,\Delta}$, we associate an integrable differential system $\mathsf{D}_{\mathsf{dif}}(V)$ in several ($= \#\Delta$) variables. We prove that this system is trivial (*i.e.* has a full set of solutions) if and only if the $G_{K,\Delta}$-representation V is de Rham. Moreover, we relate the differential module $\mathsf{D}_{\mathsf{dif}}(V)$ with overconvergent (φ, Γ)-module arising from Pal-Zábrádi theory (*cf* [20]).

Before giving a precise description of the content of this article, we make some remarks and thoughts for future developments. First of all, we note that the theory we develop here does not fit in the framework of relative p-adic Hodge theory as studied by Andreatta, Brinon ([1], [2], [3], [9]). In fact the geometric base for our objects is merely a finite discrete space (*cf* Remark 3.23). Secondly, this article should be seen as a first step towards the introduction of the multivariable periods rings $\mathsf{B}_{\mathrm{cris}}$ and B_{st}, and eventually, a step in the direction of full analogue of Berger's results via the theory of p-adic differential systems in several variables (as was foreseen in [20]).

We now describe more precisely the content of this work. In the second section we fix some notations and recall useful results. In particular we recall classical Sen theory of C-representations, where C is the completion of an algebraic closure of K, and introduce the completion C_Δ of the tensor product of the $\#\Delta$-fold tensor product of C over the maximal unramified subextension of K. In the third section we study free C_Δ-representations of finite rank of $G_{K,\Delta}$: more precisely, we develop an analogue of Sen theory in this context. Classically (*i.e.* when $\#\Delta = 1$) Sen theory is an equivalence of categories between C-representations of G_K and K_∞-representations of $\Gamma_K = \mathsf{Gal}(K_\infty/K)$, where K_∞ is the cyclotomic extension of K. Our result in the multivariable context is Theorem 3.28: there is an equivalence of categories between that of free C_Δ-representations of finite rank of $G_{K,\Delta}$ and that of free $K_{\Delta,\infty}$-representation of finite rank of $\Gamma_{K,\Delta}$, where $K_{\Delta,\infty}$ is the (non completed) $\#\Delta$-fold tensor product of K_∞ over the maximal unramified subextension of K. To the latter we can associate generalized Sen operators (describing the infinitesimal action of $\Gamma_{K,\Delta}$) and develop a Hodge-Tate theory (*cf* Corollaries 3.35 and 3.36). In the fourth section we introduce the period rings $\mathsf{B}_{\mathrm{dR},\Delta}$ and $\mathsf{B}_{\mathrm{HT},\Delta}$ and the corresponding de Rham and Hodge-Tate representations: in particular we show that there are functors D_{dR} and D_{HT} having the expected properties (Propositions 4.17, 4.18 and 4.19), in particular that being de Rham implies to be Hodge-Tate. In the fifth section we prove the multivariable analogue of the work of Fontaine in [16]: namely Sen theory for $\mathsf{B}_{\mathrm{dR},\Delta}$-representations. To do this we follow [3]: the central result (Theorem 5.11) is that the category of free $\mathsf{B}^+_{\mathrm{dR},\Delta}$-representations of finite rank of $G_{K,\Delta}$ is equivalent to that of free $\mathsf{I}^+_{\mathrm{dR},\Delta} = K_{\Delta,\infty}[\![t_\alpha]\!]_{\alpha\in\Delta}$-representations of finite rank of $\Gamma_{K,\Delta}$ (where t_α is a p-adic $2i\pi$ corresponding to the action of the factor of index α in $G_{K,\Delta}$). By inverting $\prod_{\alpha\in\Delta} t_\alpha$, we deduce the analogue for $\mathsf{B}_{\mathrm{dR},\Delta}$ (Theorem 5.12). The upshot is that we can associate a free module $\mathsf{D}_{\mathsf{dif}}(V)$ with a regular, integrable connection in $\#\Delta$ variables with coefficients in $\mathsf{I}_{\mathrm{dR},\Delta} = \mathsf{I}^+_{\mathrm{dR},\Delta}\left[\frac{1}{t_\alpha}\right]_{\alpha\in\Delta}$ to any p-adic representation V of $G_{K,\Delta}$. This is the analogue to that introduced by Fontaine in [16], and used by Berger in [5]. In particular, we show that a p-adic representation V of $G_{K,\Delta}$ is de Rham if and only if the associated module with connection $\mathsf{D}_{\mathsf{dif}}(V)$ is trivial (Proposition 5.18), and relate our construction to that of overconvergent (φ,Γ)-modules developed by Pal-Zábrádi (*cf* [20]) and Carter-Kedlaya-Zábrádi (*cf* [12]) by an analogue of [5, Corollaire 5.8] (*cf* Theorem 5.23).

Remark 1.1. There is little doubt that a general Tate-Sen formalism (such as that of [2]) does exist in the multivariable case, and that could be applied to families of multivariable representations (as for [6, §3] and [6, Proposition 5.2.1]). This said, we proceed here with Tate-Sen descent by hand (this already contains most of the necessary ideas).

2. Notations

Let K be a complete discrete valuation field of characteristic 0, with perfect residue field k of characteristic $p>0$. Fix an algebraic closure $\overline{K}$ of K and let $G_K=\mathsf{Gal}(\overline{K}/K)$. Denote by v the valuation on K normalized by $v(p)=1$. It extends uniquely to a valuation of $\overline{K}$: let C be the completion of the latter. If F is a subextension of C/K, we will denote by $\mathcal{O}_F$ (resp. $\mathfrak{m}_F$) its ring of integers (resp. its maximal ideal). By continuity, the action of G_K extends to C. Fix $\varepsilon=(\varepsilon^{(n)})_{n\in\mathbf{N}}$ a compatible system of primitive p^n-th roots of the unity (*i.e.* such that $\varepsilon^{(0)}=1$, $\varepsilon^{(1)}\neq 1$ and $(\varepsilon^{(n+1)})^p=\varepsilon^{(n)}$ for all $n\in\mathbf{N}$). For each $n\in\mathbf{N}$, put $K_n=K(\varepsilon^{(n)})$ and let $K_\infty=\bigcup\limits_{n=0}^{\infty}K_n$ be the cyclotomic extension, and $L=\widehat{K_\infty}$ its completion with respect to v. Put $H_K=\mathsf{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K=\mathsf{Gal}(K_\infty/K)$. The cyclotomic character $\chi\colon\Gamma_K\to\mathbf{Z}_p^\times$ is characterized by $\gamma\big(\varepsilon^{(n)}\big)=\big(\varepsilon^{(n)}\big)^{\chi(\gamma)}$ for all $\gamma\in\Gamma_K$: it induces an continuous isomorphism between Γ_K and an open subgroup of $\mathbf{Z}_p^\times$. We still denote χ the composite $G_K\to\Gamma_K\xrightarrow{\chi}\mathbf{Z}_p^\times$. In what follows, cohomology will always refer to *continuous* cohomology.

Using ramification estimates, Tate proved in [24] that $\mathsf{H}^i(H_K,C)=\begin{cases}L & \text{if } i=0\\ 0 & \text{if } i>0\end{cases}$ and constructed the so-called Tate's normalized traces $(R_n\colon L\to K_n)_{n\geq n_K}$ (for some integer $n_K\in\mathbf{N}$), that he used to show that $\dim_K\mathsf{H}^i(\Gamma_K,C^{H_K})=\begin{cases}1 & \text{if } i\in\{0,1\}\\ 0 & \text{if } i>1\end{cases}$, so that $\dim_K\mathsf{H}^i(G_K,C)=\begin{cases}1 & \text{if } i\in\{0,1\}\\ 0 & \text{if } i>1\end{cases}$.

Recall that Tate's normalized trace map $R_n\colon L\to K_n$ induces the map $x\mapsto\frac{1}{p^{m-n}}\mathsf{Tr}_{K_m/K_n}(x)$ on K_m for all $m\geq n_K$.

Proposition 2.1. *(cf [24, §3], [6, §3.1 & Proposition 4.1.1] and [11, Proposition 14.1.6]) These maps have the following properties:*

(i) *R_n is a K_n-linear projector onto K_n: put $X_n=\mathsf{Ker}(\mathsf{Id}-R_n)\subset L$;*

(ii) *R_n commutes to the action of Γ_K;*

(iii) $(\forall c_2\in\mathbf{R}_{>0})\,(\forall x\in L)\,v_p(R_n(x))\geq v_p(x)-c_2$ *(in particular R_n is continuous);*

(iv) $(\forall x\in L)\ \lim\limits_{n\to\infty}R_n(x)=x$*;*

(v) *for all* $c_3 > \frac{1}{p-1}$*, there exists* $n'_K \geq n_K$ *such that for all* $n \geq n'_K$ *and* $\gamma \in \Gamma_K$ *such that* $v_p(1-\chi(\gamma)) \leq n'_K$*, then* $\gamma - 1$ *is invertible on* X_n*, and for all* $x \in X_n$*, we have* $v_p((\gamma-1)^{-1}(x)) \geq v_p(x) - c_3$ *(in particular,* $\gamma - 1$ *induces an homeomorphism from* X_n *to itself).*

In what follows, we will use the previous properties with $c_2 = c_3 = 1$. This is certainly not optimal, but for technical reasons, it is much more convenient to make computations and work with $\mathcal{O}_{C_\Delta}/(p)$ (*cf infra*) rather than with quotients by elements of non integral valuation.

Let $d \in \mathbf{N}_{>0}$. Based on the work of Tate, Sen showed in [22] that the set $\mathsf{H}^1(H_K, \mathsf{GL}_d(C))$ is trivial, so that the inflation map

$$\mathsf{H}^1(\Gamma_K, \mathsf{GL}_d(L)) \to \mathsf{H}^1(G_K, \mathsf{GL}_d(C))$$

is bijective. He also proved that the natural map

$$\mathsf{H}^1(\Gamma_K, \mathsf{GL}_d(K_\infty)) \to \mathsf{H}^1(\Gamma_K, \mathsf{GL}_d(L))$$

is bijective. This means that if W is a C-representation of G_K (*cf* §3.6), there exists a K_∞-representation W_∞ of Γ_K such that $W \simeq C \otimes_{K_\infty} W_\infty$ as $C[G_K]$-modules.

As mentioned in the introduction, the first aim of this note is to generalize these results to the case where G_K is replaced by a finite power of G_K (along the lines of [20]).

Let Δ be a finite set, and put $\delta = \#\Delta$. Put $G_{K,\Delta} = \prod\limits_{\alpha\in\Delta} G_K$, $H_{K,\Delta} = \prod\limits_{\alpha\in\Delta} H_K$, and $\Gamma_{K,\Delta} = \prod\limits_{\alpha\in\Delta} \Gamma_K$. We have an exact sequence

$$1 \to H_{K,\Delta} \to G_{K,\Delta} \xrightarrow{\chi_\Delta} \Gamma_{K,\Delta} \to 1.$$

The morphism $\chi_\Delta = \prod\limits_{\alpha\in\Delta} \chi$ identifies $\Gamma_{K,\Delta}$ with an open subgroup of $(\mathbf{Z}_p^\times)^\Delta$. Il $\alpha \in \Delta$, we denote by $G_{K,\alpha}$ the image of the group homomorphim $\iota_\alpha \colon G_K \to G_{K,\Delta}$ that maps g to the element whose component of index α is g and the others are $\mathsf{Id}_{\overline{K}}$. The groups $H_{K,\alpha}$ and $\Gamma_{K,\alpha}$ are defined similarly.

Notation. (1) If $\underline{n}=(n_\alpha)_{\alpha\in\Delta}\in\mathbf{Z}^\Delta$, we define a character $\chi_\Delta^{\underline{n}}\colon G_{K,\Delta}\to\mathbf{Z}_p^\times$ by

$$\chi_\Delta^{\underline{n}}\big((g_\alpha)_{\alpha\in\Delta}\big)=\prod_{\alpha\in\Delta}\chi(g_\alpha)^{n_\alpha}.$$

This provides a $\mathbf{Z}_p$-representation $\mathbf{Z}_p(\underline{n})$ of $G_{K,\Delta}$. More generally, if M is a $\mathbf{Z}_p$-module endowed with an action of $G_{K,\Delta}$, we put $M(\underline{n})=M\otimes_{\mathbf{Z}_p}\mathbf{Z}_p(\underline{n})$, endowed with the diagonal action of $G_{K,\Delta}$.
(2) If F is a closed subextension of C/F_0 (where $F_0:=\mathsf{W}(k)\big[\frac{1}{p}\big]$), let $\mathcal{O}_{F_\Delta}$ be the p-adic completion of the tensor product $\mathcal{O}_F\otimes_{\mathsf{W}(k)}\cdots\otimes_{\mathsf{W}(k)}\mathcal{O}_F$ (where the copies of $\mathcal{O}_F$ are indexed by Δ), and $F_\Delta=\mathcal{O}_{F_\Delta}\big[\frac{1}{p}\big]$. Observe that when F/F_0 is finite, $\mathcal{O}_F$ is a free $\mathsf{W}(k)$-module of finite rank, so that the tensor product $\mathcal{O}_F\otimes_{\mathsf{W}(k)}\cdots\otimes_{\mathsf{W}(k)}\mathcal{O}_F$ is p-adically separated and complete, so that F_Δ is nothing but the tensor product $F^{\otimes\Delta}=F\otimes_{F_0}\cdots\otimes_{F_0}F$ (where the copies of F are indexed by Δ).

Remark 2.2. (1) The ring $\mathcal{O}_{F_\Delta}$ depends on k. This dependence is not indicated in the notation so as not to make it heavier.
(2) When it is not mentioned, tensor products are taken over $\mathsf{W}(k)$.

The group G_{K_Δ} naturally acts on $\mathcal{O}_{C_\Delta}$ and C_Δ. Note also that $\mathcal{O}_{L_\Delta}$ is naturally a $\mathcal{O}_{K_\infty}^{\otimes\Delta}$-algebra.

3. Multivariable classical Sen theory

3.1. The cohomology of C_Δ

Lemma 3.2. *Let $r\in\mathbf{N}_{>0}$. Then the cokernel of $\mathcal{O}_{L_\Delta}/(p^r)\to\mathsf{H}^0(H_{K,\Delta},\mathcal{O}_{C_\Delta}/(p^r))$ is killed by $\mathfrak{m}_{K_\infty}^{\otimes\Delta}$. If $i>0$, the group $\mathsf{H}^i(H_{K,\Delta},\mathcal{O}_{C_\Delta}/(p^r))$ is killed by $\mathfrak{m}_{K_\infty}^{\otimes\Delta}$.*

Proof. If $\Delta'\subset\Delta$, we denote by $\mathcal{O}_{C_{\Delta,\Delta'}}$ the p-adic completion of $\mathcal{O}_{C_{\Delta'}}\otimes_{\mathsf{W}(k)}\mathcal{O}_{L_{\Delta\setminus\Delta'}}$. In particular, we have $\mathcal{O}_{C_{\Delta,\varnothing}}=\mathcal{O}_{L_\Delta}$ and $\mathcal{O}_{C_{\Delta,\Delta}}=\mathcal{O}_{C_\Delta}$.

We proceed componentwise: let $\Delta'\subset\Delta$ and $\alpha\in\Delta\setminus\Delta'$, and consider the action of $H_{K,\alpha}$ on $\mathcal{O}_{C_{\Delta,\Delta'\cup\{\alpha\}}}/(p^r)$. The topology on the latter is discrete: we have

$$\mathsf{H}^i(H_{K,\alpha},\mathcal{O}_{C_{\Delta,\Delta'\cup\{\alpha\}}}/(p^r))=\varinjlim_F\mathsf{H}^i(H_{K,\alpha},\mathcal{O}_{F,\alpha}\otimes_{\mathsf{W}(k)}\mathcal{O}_{C_{\Delta\setminus\{\alpha\},\Delta'}}/(p^r))$$

where F runs among the finite Galois subextensions of $\overline{K}/K_\infty$. Recall that by [24, §3.2, Proposition 9], we have $\mathfrak{m}_{K_\infty}\subset\mathsf{Tr}_{F/K_\infty}(\mathcal{O}_F)$ for every such F. This implies that $\mathsf{H}^i(H_{K,\alpha}, \mathcal{O}_{F,\alpha} \otimes_{\mathsf{W}(k)} \mathcal{O}_{C_{\Delta\setminus\{\alpha\},\Delta'}}/(p^r))$ is killed by $\mathfrak{m}_{K_\infty,\alpha}$ for all F (*cf* [19, Lemma 3.1]): the same happens, for $i > 0$, to $\mathsf{H}^i(H_{K,\alpha}, \mathcal{O}_{C_{\Delta,\Delta'\cup\{\alpha\}}}/(p^r))$. If $x \in \mathsf{H}^0(H_{K,\alpha}, \mathcal{O}_{F,\alpha} \otimes_{\mathsf{W}(k)} \mathcal{O}_{C_{\Delta\setminus\{\alpha\},\Delta'}}/(p^r))$ and $\eta \in \mathfrak{m}_{K_\infty,\alpha}$, let $y \in \mathcal{O}_F$ such that $\mathsf{Tr}_{F/K_\infty}(y) = \eta$. Then $\eta x = \mathsf{Tr}_{F/K_\infty,\alpha}(xy) \in \mathcal{O}_{C_{\Delta,\Delta'}}/(p^r)$, where $\mathsf{Tr}_{F/K_\infty,\alpha}\colon \mathcal{O}_{F,\alpha} \otimes_{\mathsf{W}(k)} \mathcal{O}_{C_{\Delta\setminus\{\alpha\},\Delta'}}/(p^r)) \to \mathcal{O}_{C_{\Delta,\Delta'}}/(p^r)$ denotes the tensor product of Tr_{F/K_∞} on the factor of index α with the identity on the other factors.

The lemma follows by applying the Hochschild-Serre spectral sequence finitely many times. □

Theorem 3.3. *We have* $\mathsf{H}^i(H_{K,\Delta}, C_\Delta) = \begin{cases} L_\Delta & \textit{if } i = 0 \\ 0 & \textit{if } i > 0 \end{cases}$.

Proof. By [18, Proposition 2.7.4], there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & \varprojlim_r \frac{\mathcal{O}_{L_\Delta}}{(p^r)} & \to & \prod_{r=1}^{\infty} \frac{\mathcal{O}_{L_\Delta}}{(p^r)} & \to & \prod_{r=1}^{\infty} \frac{\mathcal{O}_{L_\Delta}}{(p^r)} & \to & \varprojlim_r{}^{(1)} \frac{\mathcal{O}_{L_\Delta}}{(p^r)} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \varprojlim_r \left(\frac{\mathcal{O}_{C_\Delta}}{(p^r)}\right)^{H_{K,\Delta}} & \to & \prod_{r=1}^{\infty} \left(\frac{\mathcal{O}_{C_\Delta}}{(p^r)}\right)^{H_{K,\Delta}} & \to & \prod_{r=1}^{\infty} \left(\frac{\mathcal{O}_{C_\Delta}}{(p^r)}\right)^{H_{K,\Delta}} & \to & \varprojlim_r{}^{(1)} \left(\frac{\mathcal{O}_{C_\Delta}}{(p^r)}\right)^{H_{K,\Delta}} & \to 0
\end{array}$$

The first three vertical maps are injective and the cokernels of those in the middle are killed by $\mathfrak{m}_{K_\infty}^{\otimes\Delta}$. This implies that the cokernel of the first vertical map is killed by $\mathfrak{m}_{K_\infty}^{\otimes\Delta}$. This implies that the cokernel of composite map

$$\mathcal{O}_{L_\Delta} \hookrightarrow \mathcal{O}_{C_\Delta}^{H_{K,\Delta}} \to \varprojlim_r (\mathcal{O}_{C_\Delta}/(p^r))^{H_{K,\Delta}}$$

is killed by $\mathfrak{m}_{K_\infty}^{\otimes\Delta}$, showing (by injectivity of the second map) that the cokernel of

$$\mathcal{O}_{L_\Delta} \to \mathsf{H}^0(H_{K,\Delta}, \mathcal{O}_{C_\Delta})$$

is killed by $\mathfrak{m}_{K_\infty}^{\otimes\Delta}$. On the other hand, the inverse system $\{\mathcal{O}_{L_\Delta}/(p^r)\}_r$ has the Mittag-Leffler property: we have $\varprojlim_r{}^{(1)} \mathcal{O}_{L_\Delta}/(p^r) = 0$. This implies that $\varprojlim_r{}^{(1)} (\mathcal{O}_{C_\Delta}/(p^r))^{H_{K,\Delta}}$ is killed by $\mathfrak{m}_{K_\infty}^{\otimes\Delta}$.

If $i>0$, we have an exact sequence (*cf* [18, Theorem 2.7.5])

$$0 \to \varprojlim_r{}^{(1)} \mathsf{H}^{i-1}(H_{K,\Delta}, \mathcal{O}_{C_\Delta}/(p^r)) \\ \to \mathsf{H}^i(H_{K,\Delta}, \mathcal{O}_{C_\Delta}) \to \varprojlim_r \mathsf{H}^i(H_{K,\Delta}, \mathcal{O}_{C_\Delta}/(p^r)) \to 0.$$

By Lemma 3.2 (and what precedes when $i=1$) the modules $\varprojlim_r{}^{(1)} \mathsf{H}^{i-1}(H_{K,\Delta}, \mathcal{O}_{C_\Delta}/(p^r))$ and $\varprojlim_r \mathsf{H}^i(H_{K,\Delta}, \mathcal{O}_{C_\Delta}/(p^r))$ are killed by $\mathfrak{m}_{K_\infty}^{\otimes\Delta}$, implying that $\mathsf{H}^i(H_\Delta, \mathcal{O}_{C_\Delta})$ is killed by $\big(\mathfrak{m}_{K_\infty}^{\otimes\Delta}\big)^2 = \mathfrak{m}_{K_\infty}^{\otimes\Delta}$.

The proposition follows by inverting p. □

Notation. If $\Delta' \subset \Delta$ and $n \in \mathbf{N}$, we denote by $\mathcal{O}_{L_{\Delta,\Delta',n}}$ the p-adic completion of $\mathcal{O}_{L_{\Delta\setminus\Delta'}} \otimes_{\mathsf{W}(k)} \mathcal{O}_{K_{n,\Delta'}}$. In particular, we have $\mathcal{O}_{L_{\Delta,\varnothing,n}} = \mathcal{O}_{K_n,\Delta}$ and $\mathcal{O}_{L_{\Delta,\Delta,n}} = \mathcal{O}_{L_\Delta}$. Put $L_{\Delta,\Delta',n} = \mathcal{O}_{L_{\Delta,\Delta',n}}\big[\frac{1}{p}\big]$. Let $n \geq n_K$: the map R_n induces a continuous map $\mathcal{O}_L \to \frac{1}{p}\mathcal{O}_{K_n}$. If $\alpha \in \Delta$, denote by $R_{n,\alpha}\colon L_\Delta \to L_{\Delta,\Delta\setminus\{\alpha\},n}$ be the map induced by the tensor product of Tate's normalized trace $R_n\colon L \to K_n$ (*cf* [24, §3]) on the factor of index α with the identity on the other factors (this makes sense by the continuity of R_n). This defines a continuous $L_{\Delta,\{\alpha\},n}$-linear projector that maps $\mathcal{O}_{L_{\Delta,\Delta',n}}$ into $\frac{1}{p}\mathcal{O}_{L_{\Delta,\Delta'\cup\{\alpha\},n}}$ (*cf* Proposition 2.1). We also put $\mathcal{O}_{L_{\Delta,\Delta',\infty}} = \bigcup_{n=0}^{\infty} \mathcal{O}_{L_{\Delta,\Delta',n}}$ and $L_{\Delta,\Delta',\infty} = \bigcup_{n=0}^{\infty} L_{\Delta,\Delta',n}$.

Theorem 3.4. *We have*

$$\mathsf{H}^i(G_{K,\Delta}, C_\Delta) \simeq \mathsf{H}^i(\Gamma_{K,\Delta}, L_\Delta) = \bigwedge^i \Big(\bigoplus_{\alpha\in\Delta} K_\Delta \log(\chi_\alpha)\Big).$$

Proof. The first isomorphism follows from the inflation-restriction exact sequence. For all $n \in \mathbf{N}$, the map $K_\Delta \to K_{n,\Delta}$ is finite étale, it is enough to prove the statement replacing K by K_n with $n \geq n_K$ (*cf* above). Then the Horschild-Serre spectral sequence reduces the proof of the second isomorphism to the equalities

$$\mathsf{H}^i(\Gamma_{K_{n,\alpha}}, L_{\Delta,\Delta',n}) = \begin{cases} L_{\Delta,\Delta'\cup\{\alpha\},n} & \text{if } i=0 \\ L_{\Delta,\Delta'\cup\{\alpha\},n}\log(\chi_\alpha) & \text{if } i=1 \\ 0 & \text{if } i>1 \end{cases}$$

for all $\alpha \in \Delta$ and $\Delta' \subset \Delta \setminus \{\alpha\}$. Working modulo p^r as we did in the proof of Lemma 3.2 and using the maps $R_{m,\alpha}$ for $m \geq n$, we deduce that the cokernels of the maps

$$\mathcal{O}_{L_{\Delta,\Delta',n}}/(p^r) \to \mathsf{H}^0(\Gamma_{K_{n,\alpha}}, L_{\Delta,\Delta',n}/(p^r))$$
$$\mathcal{O}_{L_{\Delta,\Delta',n}}/(p^r)\log(\chi_\alpha) \to \mathsf{H}^1(\Gamma_{K_{n,\alpha}}, L_{\Delta,\Delta',n}/(p^r))$$

are killed by p, as do the elements of $\mathsf{H}^i(\Gamma_{K_{n,\alpha}}, L_{\Delta,\Delta',n}/(p^r))$ if $i > 1$. Then we can pass to the limit arguing as in the proof of Proposition 3.3 to conclude that the cokernels of the maps

$$\mathcal{O}_{L_{\Delta,\Delta',n}} \to \mathsf{H}^0(\Gamma_{K_{n,\alpha}}, L_{\Delta,\Delta',n})$$
$$\mathcal{O}_{L_{\Delta,\Delta',n}}\log(\chi_\alpha) \to \mathsf{H}^1(\Gamma_{K_{n,\alpha}}, L_{\Delta,\Delta',n})$$

and the modules $\mathsf{H}^i(\Gamma_{K_{n,\alpha}}, L_{\Delta,\Delta',n})$ with $i > 1$ are killed by p. The theorem follows by inverting p. □

Theorem 3.5. *If $n \in \mathbf{N}$ and $\underline{n} \in \mathbf{Z}^\Delta \setminus \{\underline{0}\}$, we have $\mathsf{H}^i(G_{K,\Delta}, C_\Delta(\underline{n})) = 0$ for all $i \in \mathbf{N}$.*

Proof. Again, using the inflation-restriction exact sequence and the Horschild-Serre spectral sequence, we are reduced to show that if $\alpha \in \Delta$ is such that $n_\alpha \neq 0$, then the cohomology groups $\mathsf{H}^i(\Gamma_{K,\alpha}, L_\Delta(\underline{n}))$ all vanish. Let $\gamma \in \Gamma_{K,\alpha}$ be such that the closure $\overline{\langle\gamma\rangle}$ has finite index in $\Gamma_{K,\alpha}$: it is enough to show that $\mathsf{H}^i(\overline{\langle\gamma_\alpha\rangle}, L_\Delta(\underline{n})) = 0$ for all $i \in \mathbf{N}$, where $\gamma_\alpha \in \Gamma_{K,\Delta}$ is the element whose components are the identity except that of index α, which is γ. As these cohomology groups are those of the complex $L_\Delta(\underline{n}) \xrightarrow{\gamma_\alpha - 1} L_\Delta(\underline{n})$ (concentrated in degrees 0 and 1), this is equivalent to showing that $\gamma_\alpha - 1$ is bijective on $L_\Delta(\underline{n})$, *i.e.* that it is injective with cokernel killed by p^r for some $r \in \mathbf{N}$ on $\mathcal{O}_{L_\Delta}(\underline{n})$. This follows from the fact that $\gamma - 1$ is injective with cokernel killed by p^r for some $r \in \mathbf{N}$ on $\mathcal{O}_L(n_\alpha)$, since it is bijective with continuous inverse on L (*cf* Proposition 2.1 (v)). □

3.6. Sen theory for C_Δ-representations

We fix terminology and notation that will be used hereafter.

Definition. Let G be a topological group and B a topological ring endowed with a continuous action of G. A *B-representation* of G is a topological module of finite type W endowed with a continuous and semi-linear action of

G, *i.e.* such that $g(w_1+bw_2)=g(w_1)+g(b)g(w_2)$ for all $b\in B$, $w_1,w_2\in W$ and $g\in G$. We say that W is free (resp. projective) of rank d when the underlying B-module is. We denote by $\mathbf{Rep}_B(G)$ (resp. $\mathbf{Rep}_B^{\mathsf{f}}(G)$, resp. $\mathbf{Rep}_B^{\mathsf{pr}}(G)$) the category of B-representations with G-equivariant maps (resp. the full subcategory of free, resp. projective B-representations of finite rank).

Remark 3.7. If W is a free B-representation of rank d of G, and $\mathfrak{B}$ is a basis of W over B, we can denote by $U_g\in\mathsf{M}_d(B)$ the matrix of g acting on W in the basis $\mathfrak{B}$. Then $U_g\in\mathsf{GL}_d(B)$ for all $g\in G$, and the map $g\mapsto U_g$ is a continuous 1-cocycle $G\to\mathsf{GL}_d(B)$. Conversely, the data of such a cocycle endows B^d with a B-representation structure. Moreover, changing the basis precisely amounts to replace the cocycle by a cohomologous one. This means that isomorphism classes of free B-representations of rank d are in bijection with the continuous cohomology set $\mathsf{H}^1(G,\mathsf{GL}_d(B))$.

Fix $d\in\mathbf{N}_{>0}$. Let $H_0\leq H_K$ be an open normal subgroup, and put $H_\Delta=\prod\limits_{\alpha\in\Delta}H_0$. If $\Delta'\subset\Delta$, let $H_{\Delta,\Delta'}$ be the subgroup of $H_{K,\Delta}$ generated by the subgroups $\iota_\alpha(H_0)$ for $\alpha\in\Delta'$: we have $H_{\Delta,\varnothing}=\{1\}$ and $H_{\Delta,\Delta}=H_\Delta$. Note that although the groups H_Δ and $H_{\Delta,\Delta'}$ depend on H_0, we do not indicate this dependency in order not to make the notations too heavy.

Lemma 3.8. *(cf [22, Lemma 1] and [2, Lemme 2.1]) Let $U\colon H_\Delta\to\mathsf{GL}_d(C_\Delta)$ be a continuous cocycle. Let $\Delta'\subset\Delta$, $\alpha\in\Delta\setminus\Delta'$ and $m\in\mathbf{N}_{\geq 2}$ be such that $U_h=\mathrm{I}_d$ for all $h\in H_{\Delta,\Delta'}$ and $U_h\in\mathrm{I}_d+p^m\,\mathsf{M}_d\big(\mathcal{O}_{C_\Delta}^{H_{\Delta'}}\big)$ for all $h\in H_\Delta$. Then there exists $B_0\in\mathrm{I}_d+p^{m-1}\,\mathsf{M}_d\big(\mathcal{O}_{C_\Delta}^{H_{\Delta'}}\big)$ such that $B_0^{-1}U_hh(B_0)\in\mathrm{I}_d+p^{m+1}\,\mathsf{M}_d\big(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\big)$ for all $h\in\iota_\alpha(H_0)$.*

Proof. By continuity of U, there exists an open normal subgroup $H_1\leq H_0$ such that $U_h\in\mathrm{I}_d+p^{m+2}\,\mathsf{M}_d\big(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\big)$ for all $h\in\iota_\alpha(H_1)$. Let T be a complete set of representatives of $H_0/H_1\simeq\iota_\alpha(H_0)/\iota_\alpha(H_1)$: if $h\in\iota_\alpha(H_0)$ and $\tau\in T$, there exists unique $h'\in\iota_\alpha(H_1)$ and $\tau'\in T$ such that $h\tau=\tau'h'$: we have

$$U_{h\tau}=U_{\tau'h'}=U_{\tau'}\tau'(U_{h'})\in U_{\tau'}+p^{m+2}\,\mathsf{M}_d\big(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\big).$$

By [24, §3.2, Proposition 9], we can choose $c\in\mathcal{O}_C^{H_1}$ such that $\mathsf{Tr}_{H_0/H_1}(c):=\sum\limits_{\tau\in T}\tau(c)=p$. Put

$$B_0=\tfrac{1}{p}\sum_{\tau\in T}\tau(c_\alpha)U_\tau\in\mathsf{M}_d\big(C_\Delta^{H_{\Delta,\Delta'}}\big)$$

(where $c_\alpha \in \mathcal{O}_{C_\Delta}$ is the image of the tensor $1 \otimes \cdots \otimes 1 \otimes c \otimes 1 \otimes \cdots \otimes 1$, with c on the factor of index α). As $U_\tau \in \mathrm{I}_d + p^m \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$ for all $\tau \in T$, we have $B_0 \in \mathrm{I}_d + p^{m-1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$. This implies in particular that B_0 is invertible in $\mathrm{I}_d + p^{m-1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$, with $B_0^{-1} = \sum\limits_{i=0}^{\infty} (\mathrm{I}_d - B_0)^i$. Moreover, if $h \in \iota_\alpha(H_0)$, we have

$$h(B_0) = \tfrac{1}{p} \sum_{\tau \in T} h\tau(c_\alpha) h(U_\tau) = \tfrac{1}{p} \sum_{\tau \in T} h\tau(c_\alpha) U_h^{-1} U_{h\tau}$$

$$= \tfrac{1}{p} U_h^{-1} \sum_{\tau \in T} \tau' h'(c_\alpha) U_{\tau' h'} = \tfrac{1}{p} U_h^{-1} \sum_{\tau \in T} \tau'(c_\alpha) U_{\tau' h'}.$$

As $U_{\tau' h'} \in U_{\tau'} + p^{m+2} \,\mathsf{M}_d(\mathcal{O}_{C_\Delta}^{H_{\Delta'}})$, we have

$$U_h h(B_0) \in \underbrace{\tfrac{1}{p} \sum_{\tau \in T} \tau'(c_\alpha) U_{\tau'}}_{=B_0} + p^{m+1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$$

hence $B_0^{-1} U_h h(B_0) \in \mathrm{I}_d + p^{m+1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$. □

Lemma 3.9. *Under the assumptions of Lemma 3.8, there exists* $B_\alpha \in \mathrm{I}_d + p^{m-1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$ *such that the cocycle* $U \colon H_\Delta \to \mathsf{GL}_d(C_\Delta)$ *defined by* $U'_h := B_\alpha^{-1} U_h h(B_\alpha)$ *is such that* $U'_h = \mathrm{I}_d$ *for all* $h \in H_{\Delta,\Delta' \cup \{\alpha\}}$ *and* $U'_h \in \mathrm{I}_d + p^{m-1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta' \cup \{\alpha\}}}\right)$ *for all* $h \in H_\Delta$.

Proof. Lemma 3.8 produces inductively a sequence $(B_n)_{n \geq 0}$ in $\mathrm{I}_d + p^{m-1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$ and a sequence of cocycles $\left(U_n \colon H \to \mathsf{GL}_d\left(C_\Delta^{H_{\Delta,\Delta'}}\right)\right)_{n \geq 0}$ such that $B_n \in \mathrm{I}_d + p^{m+n-1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$, $U_0 = U$, $U_{n+1,h} = B_n^{-1} U_{n,h} h(B_n)$ for all $h \in H_\Delta$ and $U_{n,h} \in \mathrm{I}_d + p^{n+m} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$ for all $h \in \iota_\alpha(H_0)$ and $n \in \mathbf{N}$. The infinite product $B_\alpha = B_0 B_1 \cdots$ converges in $\mathrm{I}_d + p^{m-1} \,\mathsf{M}_d\left(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\right)$: for $h \in H_\Delta$, put $U'_h = B_\alpha^{-1} U_h h(B_\alpha)$. By construction, we have $U'_h = \mathrm{I}_d$ for all $h \in \iota_\alpha(H_0)$.
If $h \in H_{\Delta,\Delta'}$, we have $h(B_\alpha) = B_\alpha$ and $U_h = \mathrm{I}_d$ hence $U'_h = \mathrm{I}_d$. This shows that $U'_h = \mathrm{I}_d$ for all $h \in H_{\Delta,\Delta' \cup \{\alpha\}}$.
If $\beta \in \Delta \setminus (\Delta' \cup \{\alpha\})$, $h \in \iota_\alpha(H_0)$ and $\eta \in \iota_\beta(H_0)$, we have $h\eta = \eta h$, so $U'_h h(U'_\eta) = U'_\eta \eta(U'_h)$: as $U'_h = \mathrm{I}_d$, we get $h(U'_\eta) = U'_\eta$. This implies that $U'_\eta \in \mathsf{GL}_d\left(C_\Delta^{H_{\Delta,\Delta \cup \{\alpha\}}}\right)$.

As $B_\alpha \in \mathrm{I}_d + p^{m-1}\,\mathsf{M}_d\big(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\big)$ and $U_h \in \mathrm{I}_d + p^m\,\mathsf{M}_d\big(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\big)$, we have

$$U_h' = B_\alpha^{-1} U_h h(B_\alpha) \in \mathrm{I}_d + p^{m-1}\,\mathsf{M}_d\big(\mathcal{O}_{C_\Delta}^{H_{\Delta,\Delta'}}\big)$$

for all $h \in H$. $\square$

Proposition 3.10. *(cf [22, Proposition 4], [2, Proposition 2.2]) Let $U\colon H_\Delta \to \mathsf{GL}_d(C_\Delta)$ be a continuous cocycle such that $U_h \equiv \mathrm{I}_d \bmod p^r\,\mathsf{M}_d(\mathcal{O}_{C_\Delta})$ for all $h \in H_\Delta$, where $r \in \mathbf{N}_{\geq \delta+1}$. Then there exists $B \in \mathrm{I}_d + p^{r-\delta}\,\mathsf{M}_d(\mathcal{O}_{C_\Delta})$ such that $B^{-1}U_h h(B) = \mathrm{I}_d$ for all $h \in H_\Delta$. In particular U has a trivial image in $\mathsf{H}^1(H_\Delta, \mathsf{GL}_d(\mathcal{O}_{C_\Delta}))$.*

Proof. Using Lemma 3.9, this follows inductively from the case $\Delta' = \varnothing$ and $m = r$, working componentwise. $\square$

Corollary 3.11. *(cf [2, Corollaire 2.3]) The maps*

$$\varinjlim_{\substack{H \lhd H_{K,\Delta} \\ H \text{ open}}} \mathsf{H}^1(H_{K,\Delta}/H, \mathsf{GL}_d(C_\Delta^H)) \to \mathsf{H}^1(H_{K,\Delta}, \mathsf{GL}_d(C_\Delta))$$
$$\varinjlim_{\substack{H \lhd H_{K,\Delta} \\ H \text{ open}}} \mathsf{H}^1(G_{K,\Delta}/H, \mathsf{GL}_d(C_\Delta^H)) \to \mathsf{H}^1(G_{K,\Delta}, \mathsf{GL}_d(C_\Delta))$$

induced by inflation maps are bijective.

Proof. The second statement follows from the first one. Let $U\colon H_{K,\Delta} \to \mathsf{GL}_d(C_\Delta)$ be a continuous cocycle. By continuity, there exists an open normal subgroup $H \leq H_{K,\Delta}$ such that $U_h \in \mathrm{I}_d + p^{\delta+1}\,\mathsf{M}_d(\mathcal{O}_{C_\Delta})$ for all $h \in H$. Making H smaller if necessary, we can assume that $H = \prod\limits_{\alpha\in\Delta} H_0$ where $H_0 \leq H_K$ is an open normal subgroup (note that such subgroups are cofinal among open normal subgroups of $H_{K,\Delta}$). Proposition 3.10 shows that the cocycle we started with has a trivial image in $\mathsf{H}^1(H, \mathsf{GL}_d(C_\Delta))$. The inflation-restriction exact sequence of sets:

$$\{1\} \to \mathsf{H}^1\big(H_{K,\Delta}/H, \mathsf{GL}_d(C_\Delta^H)\big) \to \mathsf{H}^1\big(H_{K,\Delta}, \mathsf{GL}_d(C_\Delta)\big) \to \mathsf{H}^1\big(H, \mathsf{GL}_d(C_\Delta)\big)$$

(*cf* [23, I, §5.8]) shows that U is the induction of a unique class in $\mathsf{H}^1\big(H_{K,\Delta}/H, \mathsf{GL}_d(C_\Delta^H)\big)$. $\square$

Let $H_0 \leq H_K$ be an open normal subgroup. By [4], the field C^{H_0} is the closure of $\overline{K}^{H_0}$. The latter is a finite extension of $\overline{K}^{H_K} = K_\infty$. If x is a primitive element for $\overline{K}^{H_0}/K_\infty$, there exists $n \in \mathbf{N}$ such that x is algebraic over K_n: if we put $K' = K_n(x)$, then $\overline{K}^{H_0} = K'_\infty$, and $C^{H_0} = L' := \widehat{K'_\infty}$. In particular, if $H = \prod_{\alpha \in \Delta} H_0$, we have $C^H_\Delta = L'_\Delta$ by Theorem 3.3. In what follows, we denote by $R'_{n,\alpha}$ the generalized normalized Tate traces relative to K' (*cf* [24, §3] and Proposition 2.1).

Fix $\Delta' \subset \Delta$, $\alpha \in \Delta \setminus \Delta'$ and $n \geq n'_{K'}$. The topological decomposition $L' = K'_n \oplus X'_n$ (where $X'_n = \mathsf{Ker}(R'_n)$ gives rise to the topological decomposition

$$L'_{\Delta,\Delta'} = L'_{\Delta,\Delta'\cup\{\alpha\},n} \oplus \mathsf{Ker}(R'_{n,\alpha}).$$

Moreover, if $\gamma \in \Gamma_{K,\alpha}$ is such that $v_p(1-\chi(\gamma)) \leq n'_{K'}$, then $\gamma - 1$ is invertible on $\mathsf{Ker}(R'_{n,\alpha})$, and for all $x \in \mathsf{Ker}(R'_{n,\alpha}) \cap \mathcal{O}_{L'_{\Delta,\Delta'}}$, we have $(\gamma-1)^{-1}(x) \in \frac{1}{p}\mathcal{O}_{L'_{\Delta,\Delta'}}$ (*cf* Proposition 2.1).

Lemma 3.12. *(cf [22, Proposition 3], [6, Lemme 3.2.5]). Let $\gamma \in \Gamma_{K',\alpha}$ such that $v_p(1-\chi(\gamma)) \leq n$. Assume $B \in \mathsf{GL}_d(L'_{\Delta,\Delta'})$ and $V_1, V_2 \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right)$ are such that $\gamma(B) = V_1 B V_2$. Then $B \in \mathsf{GL}_d(L'_{\Delta,\Delta'\cup\{\alpha\},n})$.*

Proof. Put $Z = B - R'_{n,\alpha}(B) \in \mathsf{M}_d(L'_{\Delta,\Delta'})$: as $R_{n,\alpha}$ is $L'_{\Delta,\Delta'\cup\{\alpha\},n}$-linear and commutes with the action of $\Gamma_{K',\alpha}$, we have $\gamma(Z) = V_1 Z V_2$, hence

$$\gamma(Z) - Z = V_1 Z V_2 - Z = (V_1 - \mathrm{I}_d)Z + V_1 Z (V_2 - \mathrm{I}_d) - (V_1 - \mathrm{I}_d) Z (V_2 - \mathrm{I}_d).$$

If $Z \in p^a \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$, this implies that $(\gamma-1)(Z) \in p^{a+2}\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$. As Z has entries in $\mathsf{Ker}(R'_{n,\alpha})$, this implies that $Z \in p^{a+1}\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$. This shows that $Z = 0$, *i.e.* that B has entries in $L'_{\Delta,\Delta'\cup\{\alpha\},n}$. □

Lemma 3.13. *(cf [22, Lemma 3], [6, Lemme 3.2.3]). Let $a, b \in \mathbf{N}$ be such that $b \geq a > 2$ and $\gamma \in \Gamma_{K',\alpha}$ such that $v_p(1-\chi(\gamma)) \leq n$. Assume $U = \mathrm{I}_d + p^a U_1 + p^b U_2$ with $U_1 \in \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right)$ and $U_2 \in \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$. Then there exists $V \in \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$ such that $M^{-1}U\gamma(M) = \mathrm{I}_d + p^a V_1 + p^{b+1} V_2$ with $V_1 \in \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right)$ and $V_2 \in \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$, where $M = \mathrm{I}_d + p^{b-1}V$.*

Proof. We can write $U_2 = R'_{n,\alpha}(U_2) + \frac{1}{p}(1-\gamma)(V)$, where $V \in \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$ has entries in $\mathsf{Ker}(R'_{n,\alpha})$. Then we have $M^{-1} = \sum_{j=0}^{\infty} p^{j(b-1)} V^j \in \mathrm{I}_d - p^{b-1}V + p^{b+1}\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$ (because $2(b-1) \geq b+1$ since $b \geq 3$ by hypothesis), so that

$$\begin{aligned} M^{-1}U\gamma(M) &\in (\mathrm{I}_d - p^{b-1}V)U(\mathrm{I}_d + p^{b-1}\gamma(V)) + p^{b+1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right) \\ &\in U - p^{b-1}(VU - U\gamma(V)) + p^{b+1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right). \end{aligned}$$

We have $U \in \mathrm{I}_d + p^2\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$ (because $b \geq a > 2$), so that $VU - U\gamma(V) \in (1-\gamma)(V) + p^2\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$. This implies that

$$\begin{aligned} M^{-1}U\gamma(M) &\in U - p^{b-1}(1-\gamma)(V) + p^{b+1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right) \\ &\in \mathrm{I}_d + p^a U_1 + p^b\big(R'_{n,\alpha}(U_2) + \tfrac{1}{p}(1-\gamma)(V)\big) \\ &\quad - p^{b-1}(1-\gamma)(V) + p^{b+1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right) \\ &\in \mathrm{I}_d + p^a V_1 + p^{b+1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right) \end{aligned}$$

with $V_1 = U_1 + p^{b-a}R'_{n,\alpha}(U_2) \in \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right)$. □

Corollary 3.14. *(cf [22, Proposition 6], [6, Corollaire 3.2.4]). Let $a > 2$, $U \in \mathrm{I}_d + p^a\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$ and $\gamma \in \Gamma_{K',\alpha}$ such that $v_p(1-\chi(\gamma)) \leq n$. Then there exists $M \in \mathrm{I}_d + p^{a-1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$ such that $M^{-1}U\gamma(M) \in \mathrm{I}_d + p^a\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right)$.*

Proof. Using the previous lemma inductively, we can construct a sequence of matrices $(M_b)_{b\geq a}$ such that $M_b \in \mathrm{I}_d + p^{b-1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$ and

$$\begin{aligned} &(M_a M_{a+1}\cdots M_b)^{-1}U\gamma(M_a M_{a+1}\cdots M_b) \\ &\qquad\qquad \in \mathrm{I}_d + p^a\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right) + p^{b+1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right) \end{aligned}$$

for all $b \geq a$. The infinite product $\prod_{b=a}^{\infty} M_b$ converges in $\mathrm{I}_d + p^{a-1}\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'}}\right)$ and has the required property. □

By definition, we have inclusions

$$K'_{n,\Delta} = L'_{\Delta,\varnothing,n} \subset L'_{\Delta,\Delta',n} \subset L'_{\Delta,\Delta,n} = L'_{\Delta}$$

for all $n \in \mathbf{N}$, hence inclusions

$$K'_{\Delta,\infty} := \bigcup_{n=0}^{\infty} K'_{n,\Delta} = L'_{\Delta,\varnothing,\infty} \subset L'_{\Delta,\Delta',\infty} \subset L'_{\Delta}$$

for all $\Delta' \subset \Delta$. We put $\mathcal{O}_{K'_{\Delta,\infty}} = \bigcup_{n=0}^{\infty} \mathcal{O}_{K'_{n,\Delta}}$.

Corollary 3.15. *(cf [6, Proposition 3.2.6]). Let* $U\colon \Gamma_{K',\Delta} \to \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta',\infty}}\right)$ *be a continuous cocycle. Then there exists* $M \in \mathrm{I}_d + p\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta',\infty}}\right)$ *such that for all* $g \in \Gamma_{K',\Delta}$*, we have* $M^{-1}U_g g(M) \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{K'_{\Delta,\Delta'\cup\{\alpha\},\infty}}\right)$.

Proof. As $\Gamma_{K',\Delta}$ is topologically generated by finitely many elements, there exists $n \geq n'_{K'}$ such that $U_g \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta',n}}\right)$ for all $g \in \Gamma_{K',\Delta}$. Let $\gamma \in \Gamma_{K',\alpha}$ be an element of infinite order. Enlarging n if necessary, we may assume that $v_p(1-\chi(\gamma)) \leq n$. By Corollary 3.14, there exists $M \in \mathrm{I}_d + p\,\mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta',n}}\right)$ such that $M^{-1}U_\gamma\gamma_0(M) \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right)$.
For all $g \in \Gamma_{K',\Delta}$, put $U'_g = M^{-1}U_g g(M)$: this defines a cocycle $U'\colon \Gamma_{K',\Delta} \to \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta',n}}\right)$ which is cohomologous to U, and such that $U'_\gamma \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right)$. Let $g \in \Gamma_{K',\Delta}$. As $\Gamma_{K',\Delta}$ is commutative, we have $\gamma g = g\gamma$ hence $U'_\gamma\gamma(U'_g) = U'_g g(U'_\gamma)$ *i.e.* $\gamma(U'_g) = U'^{-1}_\gamma U'_g g(U'_\gamma)$. Lemma 3.12 applied with $V_1 = U'^{-1}_\gamma$ and $V_2 = g(U'_\gamma)$ (here he use the fact that $L_{\Delta,\Delta'\cup\{\alpha\},n}$ is stable by g, which follows form the commutativity of $\Gamma_{K',\Delta}$) implies that U'_g has coefficients in $L'_{\Delta,\Delta'\cup\{\alpha\},n}$, so that U' has values in $\mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_{\Delta,\Delta'\cup\{\alpha\},n}}\right)$. □

Corollary 3.16. *Let* $U\colon \Gamma_{K',\Delta} \to \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_\Delta}\right)$ *be a continuous cocycle. Then there exists* $B \in \mathrm{I}_d + p\,\mathsf{M}_d\left(\mathcal{O}_{L'_\Delta}\right)$ *such that* $B^{-1}U_g g(B) \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{K'_{\Delta,\infty}}\right)$ *for all* $g \in \Gamma_{K',\Delta}$.

Proof. This follows by using Corollary 3.15 finitely many times. □

Proposition 3.17. *(cf [6, Proposition 3.2.6]). Let* $U\colon G_{K,\Delta} \to \mathsf{GL}_g(C_\Delta)$ *be a continuous cocycle. There exist a finite subextension* K' *of* $\overline{K}/K$*, a matrix* $B \in \mathsf{GL}_d(C_\Delta)$ *such that* $B^{-1}U_g g(B) = \mathrm{I}_d$ *for all* $g \in H_{K',\Delta}$ *and* $B^{-1}U_g g(B) \in \mathsf{GL}_d\left(\mathcal{O}_{K'_\Delta}\right)$ *for all* $g \in G_{K',\Delta}$.

Proof. By continuity, there exists a finite Galois subextension K' of $\overline{K}/K$ such that, for all $g \in G_{K',\Delta}$, $U_g \in \mathrm{I}_d + p^{2+\delta} \mathsf{M}_d\left(\mathcal{O}_{C_\Delta}\right)$. Proposition 3.10 applied with $H_0 = H_{K'}$ and $r = 2+\delta$ implies the existence of $B_1 \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{C_\Delta}\right)$ such that $B_1^{-1}U_h h(B_1) = \mathrm{I}_d$ for all $h \in H_{K',\Delta}$. By construction, we have $U'_g := B_1^{-1}U_g g(B_1) = \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{C_\Delta}\right)$ for all $g \in G_{K',\Delta}$. Let $g \in G_{K,\Delta}$ and $h \in H_{K',\Delta}$. As $H_{K',\Delta} \lhd G_{K,\Delta}$, we have $h' = g^{-1}hg \in H_{K',\Delta}$, so that $U'_h h(U'_g) = U'_g g(U'_{h'})$ *i.e.* $h(U'_g) = U'_g$ (since $U'_h = U'_{h'} = \mathrm{I}_d$). As this

holds for all $h \in H_{K',\Delta}$, this implies that $U_g \in \mathsf{M}_d\left(C_\Delta^{H_{K',\Delta}}\right) = \mathsf{M}_d(L'_\Delta)$ (*cf* Theorem 3.3). This shows that in fact, we have $U'_g \in \mathsf{GL}_d(\mathcal{O}_{L'_\Delta})$ for all $g \in G_{K,\Delta}$, and that, in particular, the restriction of U' defines a continuous cocycle $U' \colon \Gamma_{K',\Delta} \to \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{L'_\Delta}\right) \subset \mathsf{GL}_d\left(\mathcal{O}_{L'_\Delta}\right)$.
By Corollary 3.16, there exist $B_2 \in \mathrm{I}_d + p\,\mathsf{M}_d\left(\mathcal{O}_{L'_\Delta}\right)$ and $n \in \mathbf{N}$ such that, for all $g \in \Gamma_{K',\Delta}$, $B_2^{-1} U'_g g(B_2) \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{K'_{n,\Delta}}\right)$. Replacing K' by K'_n, we may assume that $K'_n = K'$. Put $B = B_2 B_1 \in \mathsf{GL}_d(C_\Delta)$ and $U''_g = B^{-1} U_g g(B)$ for all $g \in G_{K,\Delta}$. By construction, we have $U''_g = \mathrm{I}_d$ for all $g \in H_{K',\Delta}$ and $U''_g \in \mathrm{I}_d + p^2 \mathsf{M}_d\left(\mathcal{O}_{K'_\Delta}\right)$ for all $g \in G_{K',\Delta}$.
Let $g \in G_{K,\Delta}$ and $\gamma \in G_{K',\Delta}$. As K'/K is Galois, we have $G_{K',\Delta} \lhd G_{K,\Delta}$, so that $\gamma' := g^{-1}\gamma g \in G_{K',\Delta}$. By the cocycle condition, we have $\gamma(U_g) = U_\gamma^{-1} U_g g(U_{\gamma'})$. A repeated application of Lemma 3.12 (for each $\alpha \in \Delta$) thus implies that $U_g \in \mathsf{GL}_d\left(\mathcal{O}_{K'_\Delta}\right)$. □

Corollary 3.18. *(cf [2, Théorème 3.1]) The natural map*

$$\varinjlim_{K'} \mathsf{H}^1(G_{K,\Delta}/H_{K',\Delta}, \mathsf{GL}_d(K'_\Delta)) \to \mathsf{H}^1(G_{K,\Delta}, \mathsf{GL}_d(C_\Delta))$$

(where K' runs among the finite Galois subextensions of $\overline{K}/K$) induced by inflation maps is bijective.

Proof. Surjectivity is nothing but Proposition 3.17: it remains to prove the injectivity. Let K' be a finite subextension of $\overline{K}/K$ and $U, U' \colon G_{K,\Delta}/H_{K',\Delta} \to \mathsf{GL}_g(K'_\Delta)$ be two continuous cocycles that induce cohomologous cocycles $G_{K,\Delta} \to \mathsf{GL}_d(C_\Delta)$. This means that there exists $B \in \mathsf{GL}_d(C_\Delta)$ such that $U'_g = B^{-1} U_g g(B)$ for all $g \in G_{K,\Delta}$. By continuity, we may enlarge K' and assume that $U_g, U'_g \in \mathrm{I}_d + p^2 \mathsf{M}_d(\mathcal{O}_{K',\Delta})$ for all $g \in G_{K',\Delta}$. If $g \in H_{K',\Delta}$, we have $U_g, U'_g = \mathrm{I}_d$, so that $g(B) = B$: this shows that $B \in \mathsf{GL}_d(L'_\Delta)$, where $L' = C^{H_{K'}}$. Then we have $\gamma(B) = U_\gamma^{-1} B U'_\gamma$ for all $\gamma \in G_{K,\Delta}/H_{K',\Delta}$. Applying Lemma 3.12 finitely many times (for each $\alpha \in \Delta$) shows that $B \in \mathsf{GL}_d(K'_{\Delta,\infty})$. Replacing K' by K'_n for $n \in \mathbf{N}$ large enough implies that U and U' are cohomologous as cocycles with values in $\mathsf{GL}_d(K'_\Delta)$, proving the injectivity. □

We now refine the previous statement by translating it in terms of C_Δ-representations of $G_{K,\Delta}$.

Theorem 3.19. *(cf [22, Theorem 3]). Let W be a free C_Δ-representation of $G_{K,\Delta}$ of rank d. Then there exists a free $K_{\Delta,\infty}$-representation Y of $\Gamma_{K,\Delta}$ of rank d and such that $W \simeq C_\Delta \otimes_{K_{\Delta,\infty}} Y$ (as C_Δ-representations of $G_{K,\Delta}$).*

Proof. Corollary 3.18 implies that there exists a finite Galois subextension K' of $\overline{K}/K$ and a free K'_Δ-representation of rank d of $G_{K,\Delta}/H_{K',\Delta}$ such that $W \simeq C_\Delta \otimes_{K'_\Delta} W'$ as C_Δ-representations of $G_{K,\Delta}$. Enlarging K' if necessary, we may furthermore assume that K'/F_0 is Galois. By restriction, the group $\mathsf{Gal}(K'_\infty/K_\infty)$ identifies with a subgroup $G = \mathsf{Gal}(K'/K' \cap K_\infty)$ of $\mathsf{Gal}(K'/F_0)$. Put $F = K'^G = K' \cap K_\infty$. Note that G is the kernel of the map $\mathsf{Gal}(K'/F_0) \to \Gamma_K$ induced by the restriction to K_∞: this implies that F/F_0 is Galois. If $\Delta' \subset \Delta$ and $\alpha \in \Delta \setminus \Delta'$, the finite Galois extension $F \to K'$ induces a finite étale extension

$$F_{\Delta' \cup \{\alpha\}} \otimes_K K'_{\Delta \setminus (\Delta' \cup \{\alpha\})} \to F_{\Delta'} \otimes_K K'_{\Delta \setminus \Delta'}$$

with group G_α. By Galois descent, if W_α is a rank d projective $F_{\Delta'} \otimes_K K'_{\Delta \setminus \Delta'}$-representation of G_α, then $W_\alpha^{G_\alpha}$ is a rank d projective $F_{\Delta' \cup \{\alpha\}} \otimes_K K'_{\Delta \setminus (\Delta' \cup \{\alpha\})}$-module of finite rank and the natural map

$$(F_{\Delta'} \otimes_K K'_{\Delta \setminus \Delta'}) \otimes_{F_{\Delta' \cup \{\alpha\}} \otimes_K K'_{\Delta \setminus (\Delta' \cup \{\alpha\})}} W_\alpha^{G_\alpha} \to W_\alpha$$

is an isomorphism. Starting from W' and applying what precedes for each $\alpha \in \Delta$ implies that W'^{G_Δ} is a projective F_Δ-module of rank d, and that the map

$$K'_\Delta \otimes_{F_\Delta} W'^{G_\Delta} \to W'$$

is an isomorphism. As F/F_0 is finite and Galois, F_Δ is a finite product of copies of F (indexed by $\mathsf{Gal}(F/F_0)^{\delta-1}$), so that a projective F_Δ-module is necessarily free of rank d (the dimension over F of all its localizations is d, since it is free of rank d after tensoring with C_Δ). Then we have $G_{K,\Delta}$-equivariant isomorphisms

$$W \simeq C_\Delta \otimes_{K'_\Delta} W' \simeq C_\Delta \otimes_{F_\Delta} W'^{G_\Delta} \simeq C_\Delta \otimes_{K_{\Delta,\infty}} Y$$

where $Y = K_{\Delta,\infty} \otimes_{F_\Delta} W'^{G_\Delta}$ is a $K_{\Delta,\infty}$-representation of $\Gamma_{K,\infty}$ which is free of rank d. □

Corollary 3.20. *If W is a C_Δ-representation of $G_{K,\Delta}$, then $W^{H_{K,\Delta}}$ is a free L_Δ-representation of $\Gamma_{K,\Delta}$ of rank d, and the natural map*

$$C_\Delta \otimes_{L_\Delta} W^{H_{K,\Delta}} \to W$$

is a $G_{K,\Delta}$-equivariant isomorphism.

Proof. By Theorem 3.19, we may assume that $W = C_\Delta \otimes_{K_{\infty,\Delta}} Y$ where Y ia a free $K_{\infty,\Delta}$-representation of $\Gamma_{K,\Delta}$ of rank d. Then $W^{H_{K,\Delta}} = L_\Delta \otimes_{K_{\infty,\Delta}} Y$ is a free L_Δ-representation of $\Gamma_{K,\Delta}$ of rank d, and the natural map $C_\Delta \otimes_{L_\Delta} W^{H_{K,\Delta}} \to W$ is a $G_{K,\Delta}$-equivariant isomorphism. □

Corollary 3.21. *(cf [22, Theorem 1]) The natural map*

$$\mathsf{H}^1(\Gamma_{K,\Delta}, \mathsf{GL}_d(K_{\Delta,\infty})) \to \mathsf{H}^1(G_{K,\Delta}, \mathsf{GL}_d(C_\Delta))$$

is bijective.

Proof. Here again the surjectivity follows from Theorem 3.19, and the injectivity is proved exactly as in the proof of Corollary 3.18. □

Proposition 3.22. *If $\mathcal{E} \subset L_\Delta$ is a sub-K_Δ-module of finite type stable by $\Gamma_{K,\Delta}$, then $\mathcal{E} \subset K_{\Delta,\infty}$ (more precisely, there exists $n \in \mathbf{N}$ such that $\mathcal{E} \subset K_{n,\Delta}$).*

Proof. Enlarging K, we may assume that K/F_0 is Galois, with group G_{K/F_0}. The map

$$\begin{aligned} K \otimes_{F_0} K &\to K^{G_{K/F_0}} \\ x \otimes y &\mapsto (x\sigma(y))_{\sigma \in G_{K/F_0}} \end{aligned}$$

is an isomorphism of K-algebras (with the left structure on the LHS, and the diagonal structure on the RHS). Fix an ordering $\alpha_1 < \alpha_2 < \cdots < \alpha_\delta$ of Δ: by induction, what precedes provides an isomorphism of K-algebras

$$\iota \colon K_\Delta \xrightarrow{\sim} K^{G_{K/F_0}^{\delta-1}}$$

where the component of index $\underline{\sigma} = (\sigma_2, \ldots, \sigma_\delta) \in G_{K/F_0}^{\delta-1}$ of $\iota(x_1 \otimes \cdots \otimes x_\delta)$ is given by

$$x_1\sigma_2(x_2\sigma_3(x_3 \cdots x_{\delta-1}\sigma_\delta(x_\delta)\cdots)$$

(here the K-algebra structure on the LHS is through the factor of index α_1 and that on the RHS is the diagonal one). For each $\underline{\sigma} \in G_{K/F_0}^{\delta-1}$, we thus have a surjective morphism of K-algebras $\iota_{\underline{\sigma}} \colon K_\Delta \to K$ corresponding to the projection onto the factor of index $\underline{\sigma}$. Similarly, we have an injective map $\iota_\infty \colon K_{\Delta,\infty} \to K_\infty^{G_{K_\infty/F_0}^{\delta-1}}$, that extends into an injective map

$\widehat{\iota}_\infty\colon L_\Delta \to L^{G^{\delta-1}_{K_\infty/F_0}}$ (because ι_∞ maps $\mathcal{O}_{K_{\Delta,\infty}}$ into $\mathcal{O}_L^{G^{\delta-1}_{K_\infty/F_0}}$ which is p-adically separated and complete). Let $\underline{\sigma} \in G^{\delta-1}_{K/F_0}$: the map $\iota_{\underline{\sigma}}$ is a localization, so that the natural map

$$\widehat{\iota}_{\infty,\underline{\sigma}}\colon K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} L_\Delta \to \prod_{\substack{\underline{\gamma}\in G^{\delta-1}_{K_\infty/F_0} \\ \underline{\gamma}\mapsto\underline{\sigma}}} L$$

is injective. Note that if $\underline{g} = (g_1, \ldots, g_\delta) \in \Gamma_{K,\Delta}$ and $x_1, \ldots, x_\delta \in K_\infty$, we have

$$\begin{aligned}
&\iota_\infty\big(\underline{g}(x_1 \otimes \cdots \otimes x_\delta)\big) = \iota_\infty\big(g_1(x_1) \otimes \cdots \otimes g_\delta(x_\delta)\big) \\
&= \big(g_1(x_1)\gamma_2(g_2(x_2)\gamma_3(g_3(x_3)\cdots g_{\delta-1}(x_{\delta-1})\gamma_\delta(g_\delta(x_\delta)))\cdots)\big)_{\underline{\gamma}\in G^{\delta-1}_{K_\infty/F_0}} \\
&= \Big(g_1\Big(x_1\big(g_1^{-1}\gamma_2 g_2\big)\big(x_2\big(g_2^{-1}\gamma_3 g_3\big)\big(x_3\cdots\big(g_{\delta-1}^{-1}\gamma_\delta g_\delta\big)(x_\delta)\cdots\big)\big)\Big)\Big)_{\underline{\gamma}\in G^{\delta-1}_{K_\infty/F_0}}.
\end{aligned}$$

This shows that $\widehat{\iota}_{\infty,\underline{\sigma}}$ is $\Gamma_{K,\Delta}$-equivariant when the $\prod_{\substack{\underline{\gamma}\in G^{\delta-1}_{K_\infty/F_0} \\ \underline{\gamma}\mapsto\underline{\sigma}}} L$ is equipped with the action given by

$$\underline{g} \cdot (x_{\underline{\gamma}})_{\underline{\gamma}} = \big(g_1(x_{\underline{g}\cdot\underline{\gamma}})\big)_{\underline{\gamma}},$$

where $\underline{g} \cdot \underline{\gamma} = (g_1^{-1}\gamma_2 g_2, g_2^{-1}\gamma_3 g_3, \ldots, g_{\delta-1}^{-1}\gamma_\delta g_\delta) \in G^{\delta-1}_{K/F_0}$ if $\underline{\gamma} = (\gamma_2, \ldots, \gamma_\delta)$ (note that this indeed maps to $\underline{\sigma}$ if $\underline{\gamma}$ does).

By hypothesis, the localization $\mathcal{E}_{\underline{\sigma}} := K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} \mathcal{E}$ is a finite dimensional sub-K-vector space of $K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} L_\Delta$ that is stable under the action of $\Gamma_{K,\Delta}$. If $\underline{\gamma}_0 = (\gamma_2, \ldots, \gamma_\delta) \in G^{\delta-1}_{K_\infty/F_0}$, the projection $\mathrm{pr}_{\underline{\gamma}_0} \circ \widehat{\iota}_{\infty,\underline{\sigma}}\colon K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} L_\Delta \to L$ onto the factor of index $\underline{\gamma}_0$ maps $\mathcal{E}_{\underline{\sigma}}$ onto a finite dimensional sub-K-vector space $\mathcal{E}_{\underline{\gamma}_0}$ of L. Moreover, if $g \in \Gamma_K$, define the element $\underline{g} = (g_1, \ldots, g_\delta) \in \Gamma_{K,\Delta}$ by $g_1 = g$ and $g_i = \gamma_i^{-1} g_{i-1} \gamma_i$ for all $i \in \{2, \ldots, \delta\}$ (we have indeed $\gamma_i^{-1} g_{i-1}\gamma_i \in \Gamma_K$ since Γ_K is normal in G_{K_∞/F_0} because K/F_0 is Galois). By construction, we have $\underline{g} \cdot \underline{\gamma}_0 = \underline{\gamma}_0$, and the component of index $\underline{\gamma}_0$ in

$$\underline{g}\big((x_{\underline{\gamma}})_{\underline{\gamma}}\big)$$

is precisely $g(x_{\underline{\gamma}_0})$ for all $(x_{\underline{\gamma}})_{\underline{\gamma}} \in \prod_{\substack{\underline{\gamma}\in G^{\delta-1}_{K_\infty/F_0} \\ \underline{\gamma}\mapsto\underline{\sigma}}} L$. As $\mathcal{E}_{\underline{\sigma}}$ is stable under $\Gamma_{K,\Delta}$, this implies in particular that $\mathcal{E}_{\underline{\gamma}_0}$ is stable under Γ_K. By [22, Proposition 3], this implies that there exists an integer $n_{\underline{\sigma}}$ such that $\mathcal{E}_{\underline{\gamma}_0} \subset K_{n_{\underline{\sigma}}}$.

As $\Gamma_K^{\delta-1} \subset \Gamma_{K,\Delta}$ acts transitively on the set of those elements $\underline{\gamma} \in G_{K_\infty/F_0}^{\delta-1}$ mapping to $\underline{\sigma}$ (by the action given by $(g_2,\ldots,g_\delta)\cdot(\gamma_2,\ldots,\gamma_\delta) = (\gamma_2 g_2, g_2^{-1}\gamma_3 g_3,\ldots,g_{\delta-1}^{-1}\gamma_\delta g_\delta))$, and as $\underline{g}\cdot\big((x_{\underline{\gamma}})_{\underline{\gamma}}\big) = \big(x_{\underline{g}\cdot\underline{\gamma}}\big)_{\underline{\gamma}}$ when $g_1 = \mathsf{Id}_{K_\infty}$, the stability of $\mathcal{E}_{\underline{\sigma}}$ under $\Gamma_{K,\Delta}$ implies that the projection $\mathrm{pr}_{\underline{\gamma}} \circ \widehat{\iota}_{\infty,\underline{\sigma}} \colon K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} L_\Delta \to L$ onto the factor of index $\underline{\gamma}$ maps $\mathcal{E}_{\underline{\sigma}}$ into $K_{n_{\underline{\sigma}}}$ for *all* $\underline{\gamma} \in G_{K_\infty/F_0}^{\delta-1}$ mapping to $\underline{\sigma}$.

The injectivity and equivariance of $\widehat{\iota}_{\infty,\underline{\sigma}}$ imply that $\mathcal{E}_{\underline{\sigma}}$ is fixed by $\Gamma_{K_{n_{\underline{\sigma}}},\Delta}$. Now if we call n the maximum of those $n_{\underline{\sigma}}$ (there are finitely many of these, since $G_{K/F_0}^{\delta-1}$ is finite), this shows that all the localizations $K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} \mathcal{E}$ are invariants under $\Gamma_{K_n,\Delta}$: the same holds for $\mathcal{E}$. □

Remark 3.23. The previous proposition shows that our geometric setting is that of a finite discrete space, corresponding to a finite product of fields: there are finitely many finite extensions E_i/K such that $K_\Delta \simeq \prod\limits_{i\in I} E_i$. The data of a K_Δ-module M is thus equivalent to that of the collection of its localizations $(E_i \times_{K_\Delta} M)_{i\in I}$. In particular, the K_Δ-module M is free of rank d if and only if $\dim_{E_i}(E_i \times_{K_\Delta} M) = d$ for all $i \in I$.

Definition. Let X be a L_Δ-representation of $\Gamma_{K,\Delta}$. An element $x \in X$ is said to be K_Δ-finite if its orbit under $\Gamma_{K,\Delta}$ generates a K_Δ-module of finite type in X. We denote by X_{f} the subset of elements elements that K_Δ-finite in X. In other words, X_{f} is the union of all sub-K_Δ-modules of X that are of finite type and stable by $\Gamma_{K,\Delta}$. Note that X_{f} is a sub-K_Δ-module of X, and that it is stable under $\Gamma_{K,\Delta}$.

Corollary 3.24. *If $X \in \mathbf{Rep}_{L_\Delta}(\Gamma_{K,\Delta})$ is free of rank d, then X_{f} is a free $K_{\Delta,\infty}$-module of rank d, and the natural map*

$$L_\Delta \otimes_{K_{\Delta,\infty}} X_{\mathsf{f}} \to X$$

is a $\Gamma_{K,\Delta}$-equivariant isomorphism.

Proof. By Corollary 3.21, we can find a L_Δ-basis $\mathfrak{B} = (e_1,\ldots,e_d)$ of X such that $Y := \bigoplus\limits_{i=1}^{d} K_{\Delta,\infty} e_i$ is stable by $\Gamma_{K,\Delta}$ in X. By construction, the natural map $L_\Delta \otimes_{K_{\Delta,\infty}} Y \to X$ is a $\Gamma_{K,\Delta}$-equivariant isomorphism. We have to show that $Y = X_{\mathsf{f}}$. The action of $\Gamma_{K,\Delta}$ on Y is described in the basis $\mathfrak{B}$ by a continuous cocycle $U \colon \Gamma_{K,\Delta} \to \mathsf{GL}_d(K_{\Delta,\infty})$: there exists $m \in \mathbf{N}$ such that U has values in $K_{m,\Delta}$. This implies that $Y_m := \bigoplus\limits_{i=1}^{d} K_{m,\Delta} e_i$ is stable under the action of $\Gamma_{K,\Delta}$, so that $Y_m \subset X_{\mathfrak{f}}$. As $X_{\mathfrak{f}}$ is a $K_{\infty,\Delta}$-module, this implies that $Y \subset X_{\mathfrak{f}}$. Conversely, let $x \in X_{\mathfrak{f}}$. As $x \in X$, we can write uniquely $x = \sum\limits_{i=1}^{d} \lambda_i e_i$ where $\lambda_1, \ldots, \lambda_d \in L_\Delta$. Let $\mathcal{E}$ be the sub-$K_{m,\Delta}$-module of L_Δ generated by the coordinates in $\mathfrak{B}$ of all the elements $g(x)$ with $g \in \Gamma_{K,\Delta}$. As $x \in X_{\mathfrak{f}}$, the $K_{m,\Delta}$-module $\mathcal{E}$ is of finite type, and stable under the action of $\Gamma_{K,\Delta}$ by definition (because the coordinates of $g(x)$ in $\mathfrak{B}$ are given by the matrix product $U_g \begin{pmatrix} g(\lambda_1) \\ \vdots \\ g(\lambda_d) \end{pmatrix}$ and $U_g \in \mathsf{GL}_d(K_{m,\Delta})$). By Proposition 3.22, we have $\mathcal{E} \subset K_{\infty,\Delta}$, hence $x \in Y$. □

Proposition 3.25. *Let K' be a finite extension of K in $\overline{K}$. Recall that $L = \widehat{K_\infty}$ and put $L' = \widehat{K'_\infty}$. The extensions $K_{\Delta,\infty} \to K'_{\Delta,\infty}$ and $L_\Delta \to L'_\Delta$ are finite étale, and Galois with group $\mathsf{Gal}(K'_\infty/K_\infty)^\Delta$ if K'_∞/K_∞ is Galois. Moreover, we have $K'_{\Delta,\infty} \otimes_{K_{\Delta,\infty}} L_\Delta \overset{\sim}{\to} L'_\Delta$.*

Proof. • If $\Delta' \subset \Delta$ and $\alpha \in \Delta \setminus \Delta'$, the finite separable extension $K_n \to K'_n$ tensored with $K_n'^{\otimes \Delta'} \otimes_F K_n^{\otimes(\Delta \setminus (\Delta' \cup \{\alpha\}))}$ over F provides a finite étale map

$$K_n'^{(\Delta \cup \{\alpha\})} \otimes_F K_n^{\otimes(\Delta \setminus (\Delta' \cup \{\alpha\}))}$$

for all $n \in \mathbf{N}$. the latter is Galois with group $\mathsf{Gal}(K'_\infty/K_\infty)$ when $n \gg 0$ if K'_∞/K_∞ is Galois. The composition of all these maps thus provides a finite étale map

$$K_n^{\otimes \Delta} \to K_n'^{\otimes \Delta}$$

which is Galois with group $\mathsf{Gal}(K'_\infty/K_\infty)^\Delta$ when $n \gg 0$ if K'_∞/K_∞ is Galois. Put together, this shows that the map $K_{\Delta,\infty} \to K'_{\Delta,\infty}$ is finite étale, and Galois with group $\mathsf{Gal}(K'_\infty/K_\infty)^\Delta$ if K'_∞/K_∞ is Galois.

• There exists $n_0 \in \mathbf{N}$ such that $n \geq n_0 \Rightarrow [K'_n : K_n] = [K'_\infty : K_\infty]$, so that $K'_n \otimes_{K_n} K_\infty \overset{\sim}{\to} K'_\infty$. By [1, Corollary 3.10], making n_0 larger, we may assume that the cokernel of the map $\mathcal{O}_{K'_n} \otimes_{\mathcal{O}_{K_n}} \mathcal{O}_{K_\infty} \to \mathcal{O}_{K'_\infty}$ is killed by p whenever $n \geq n_0$ (we could replace p by any element on positive valuation in $\mathcal{O}_{K_\infty}$, but we will not use this). This implies that for $n \geq n_0$, there is an exact sequence

$$0 \to \mathcal{O}_{K'_n}^{\otimes \Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes \Delta}} \mathcal{O}_{K_\infty}^{\otimes \Delta} \to \mathcal{O}_{K'_\infty}^{\otimes \Delta} \to T \to 0$$

where T is a group killed by p. If $r \in \mathbf{N}_{>0}$, we deduce the exact sequence

$$\begin{aligned} \mathsf{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/p^r\,\mathbf{Z}, \mathcal{O}_{K'_\infty}^{\otimes \Delta}) &\to \mathsf{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/p^r\,\mathbf{Z}, T) \\ &\to \mathcal{O}_{K'_n}^{\otimes \Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes \Delta}} \mathcal{O}_{K_\infty}^{\otimes \Delta}/(p^r) \to \mathcal{O}_{K'_\infty}^{\otimes \Delta}/(p^r) \to T/(p^r) \to 0. \end{aligned}$$

As $\mathcal{O}_{K'_\infty}^{\otimes\Delta}$ has no p-torsion (resp. T is killed by p), we have $\mathsf{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/p^r\mathbf{Z}, \mathcal{O}_{K'_\infty}^{\otimes\Delta}) = \{0\}$ (resp. $T/(p^r) = T$ and $\mathsf{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/p^r\mathbf{Z}, T) \simeq T$), hence an exact sequence

$$0 \to T \to \mathcal{O}_{K'_n}^{\otimes\Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes\Delta}} \mathcal{O}_{K_\infty}^{\otimes\Delta}/(p^r) \xrightarrow{f_r} \mathcal{O}_{K'_\infty}^{\otimes\Delta}/(p^r) \to T \to 0.$$

It splits into two exact sequences

$$0 \to T \to \mathcal{O}_{K'_n}^{\otimes\Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes\Delta}} \mathcal{O}_{K_\infty}^{\otimes\Delta}/(p^r) \to \mathsf{Im}(f_r) \to 0$$

$$0 \to \mathsf{Im}(f_r) \to \mathcal{O}_{K'_\infty}^{\otimes\Delta}/(p^r) \to T \to 0.$$

Passing to inverse limits gives exact sequences

$$0 \to T \to \varprojlim_r \mathcal{O}_{K'_n}^{\otimes\Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes\Delta}} \mathcal{O}_{K_\infty}^{\otimes\Delta}/(p^r) \to \varprojlim_r \mathsf{Im}(f_r) \to 0$$

$$0 \to \varprojlim_r \mathsf{Im}(f_r) \to \mathcal{O}_{L'_\Delta} \to T$$

(the exactness on the right in the first sequence follows from the fact that the constant inverse system $(T)_{r\geq 1}$ has the Mittag-Leffler property). As $\mathcal{O}_{K'_n}$ is free over $\mathcal{O}_{K_n}$, so is $\mathcal{O}_{K'_n}^{\otimes\Delta}$ over $\mathcal{O}_{K_n}^{\otimes\Delta}$: this implies that

$$\varprojlim_r \mathcal{O}_{K'_n}^{\otimes\Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes\Delta}} \mathcal{O}_{K_\infty}^{\otimes\Delta}/(p^r) \simeq \mathcal{O}_{K'_n}^{\otimes\Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes\Delta}} \varprojlim_r \mathcal{O}_{K_\infty}^{\otimes\Delta}/(p^r) = \mathcal{O}_{K'_n}^{\otimes\Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes\Delta}} \mathcal{O}_{L_\Delta}.$$

The previous exact sequences thus provide an exact sequence

$$0 \to T \to \mathcal{O}_{K'_n}^{\otimes\Delta} \otimes_{\mathcal{O}_{K_n}^{\otimes\Delta}} \mathcal{O}_{L_\Delta} \to \mathcal{O}_{L'_\Delta} \to T.$$

Inverting p gives an isomorphism

$$K_n'^{\otimes\Delta} \otimes_{K_n^{\otimes\Delta}} L_\Delta \xrightarrow{\sim} L'_\Delta,$$

hence an isomorphism $K'_{\Delta,\infty} \otimes_{K_{\Delta,\infty}} L_\Delta \xrightarrow{\sim} L'_\Delta$, showing the last assertion. The statements on the map $L_\Delta \to L'_\Delta$ follow. □

Proposition 3.26. *The $K_{\Delta,\infty}$-algebra L_Δ is faithfully flat.*

Proof. • Assume that K/F_0 is Galois: so is K_n/F_0 for all $n \in \mathbf{N}$. Recall that the choice of an ordering on Δ provides an isomorphism $K_{n,\Delta} \simeq K_n^{G_{K_n/F_0}^{\delta-1}}$

(where $G_{K_n/F_0} = \mathsf{Gal}(K_n/F_0)$): the localizations of $K_{n,\Delta}$ at maximal ideals are given by the projections $\iota_{\underline{\sigma}}\colon K_{n,\Delta} \to K_n$ indexed by the elements $\underline{\sigma} \in G_{K_n/F_0}^{\delta-1}$. The corresponding localization $K_n \to K_{n,\iota_{\underline{\sigma}}} \otimes_{K_{n,\Delta}} L_\Delta$ is flat (because K_n is a field!): this proves that the map $K_{n,\Delta} \to L_\Delta$ is faithfully flat.

Let $I \subset K_{\Delta,\infty}$ be a nonzero ideal. For each $n \in \mathbf{N}$, put $I_n = I \cap K_{n,\Delta}$ (where $K_{n,\Delta}$ is seen as a subring of $K_{\Delta,\infty}$). By flatness, the natural map $I_n \otimes_{K_{n,\Delta}} L_\Delta \to L_\Delta$ is injective for all $n \in \mathbf{N}$. The commutative diagram

$$\begin{tikzcd} I_n \otimes_{K_{n,\Delta}} L_\Delta \arrow[d, "i_n"'] \arrow[dr, hook] & \\ & L_\Delta \\ I_{n+1} \otimes_{K_{n+1,\Delta}} L_\Delta \arrow[ur, hook] & \end{tikzcd}$$

thus implies that the natural map $i_n\colon I_n \otimes_{K_{n,\Delta}} L_\Delta \to I_{n+1} \otimes_{K_{n+1,\Delta}} L_\Delta$ is injective. Passing to the inductive limit, this implies that the following composite of the natural maps

$$\varinjlim_n (I_n \otimes_{K_{n,\Delta}} L_\Delta) \to I \otimes_{K_{\Delta,\infty}} L_\Delta \to L_\Delta$$

is injective. As the first map is surjective, it is an isomorphism, so that the natural map $I \otimes_{K_{\Delta,\infty}} L_\Delta \to L_\Delta$ is injective. This proves the flatness in this case.

As seen in the proof of Proposition 3.22, if $\underline{\sigma} \in G_{K/F_0}^{\delta-1}$, there is an injection $\widehat{\iota}_{\infty,\underline{\sigma}}\colon K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} L_\Delta \to \prod\limits_{\substack{\underline{\gamma} \in G_{K_\infty/F_0}^{\delta-1} \\ \underline{\gamma} \mapsto \underline{\sigma}}} L$. It inserts in the commutative diagram

$$\begin{tikzcd} K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} K_{\Delta,\infty} \arrow[r] \arrow[d, hook] & \prod\limits_{\substack{\underline{\gamma} \in G_{K_\infty/F_0}^{\delta-1} \\ \underline{\gamma} \mapsto \underline{\sigma}}} K_\infty \arrow[d, hook] \\ K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} L_\Delta \arrow[r, "\widehat{\iota}_{\infty,\underline{\sigma}}"] & \prod\limits_{\substack{\underline{\gamma} \in G_{K_\infty/F_0}^{\delta-1} \\ \underline{\gamma} \mapsto \underline{\sigma}}} L \end{tikzcd}.$$

As the top horizontal map is faithful (because its components are precisely the localizations of $K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} K_{\Delta,\infty}$ at its maximal ideals), and as the right vertical map is faithful, so is the composite

$$K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} K_{\Delta,\infty} \to K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} L_\Delta \to \prod_{\substack{\underline{\gamma} \in G^{\delta-1}_{K_\infty/F_0} \\ \underline{\gamma} \mapsto \underline{\sigma}}} L$$

hence $K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} K_{\Delta,\infty} \to K_{\iota_{\underline{\sigma}}} \otimes_{K_\Delta} L_\Delta$ is faithful as well. As this holds for all $\underline{\sigma} \in G^{\delta-1}_{K/F_0}$, this implies that $K_{\Delta,\infty} \to L_\Delta$ is faithful.

• In general, let $K' \subset \overline{K}$ be the Galois closure of K/F_0. By what precedes, the map $K'_{\Delta,\infty} \to L'_\Delta$ is faithfully flat. Moreover, $K'_{\Delta,\infty}$ is free as a $K_{\Delta,\infty}$-module, so that $K_{\Delta,\infty} \to L'_\Delta$ is flat as well. Let $I \subset K_{\Delta,\infty}$ be a nonzero ideal. The natural map $I \otimes_{K_{\Delta,\infty}} L'_\Delta \to L'_\Delta$ is thus injective. On the other hand, L'_Δ is free over L_Δ since $K'_{\Delta,\infty}$ is over $K_{\Delta,\infty}$ and $K'_{\Delta,\infty} \otimes_{K_{\Delta,\infty}} L_\Delta \xrightarrow{\sim} L'_\Delta$ by Proposition 3.25. This implies that $I \otimes_{K_{\Delta,\infty}} L_\Delta \to I \otimes_{K_{\Delta,\infty}} L'_\Delta$ is injective, so that the composite map $I \otimes_{K_{\Delta,\infty}} L_\Delta \to L'_\Delta$ is injective. As it factors through L_Δ, this implies that $I \otimes_{K_{\Delta,\infty}} L_\Delta \to L_\Delta$ is injective, showing the flatness of $K_{\Delta,\infty} \to L_\Delta$.

If M is a $K_{\Delta,\infty}$-module such that $M \otimes_{K_{\Delta,\infty}} L_\Delta = 0$, we have

$$(M \otimes_{K_{\Delta,\infty}} K'_{\Delta,\infty}) \otimes_{K'_{\Delta,\infty}} L'_\Delta \simeq M \otimes_{K_{\Delta,\infty}} L'_\Delta \simeq (M \otimes_{K_{\Delta,\infty}} L_\Delta) \otimes_{L_\Delta} L'_\Delta = 0$$

so that the faithfulness of L'_Δ over $K'_{\Delta,\infty}$ implies that $M \otimes_{K_{\Delta,\infty}} K'_{\Delta,\infty} = 0$, hence $M = 0$ since $K'_{\Delta,\infty}$ is free over $K_{\Delta,\infty}$. □

Corollary 3.27. *(cf [3, Lemme 3.15]) Let X_1 and X_2 be free L_Δ-representations of finite rank of $\Gamma_{K,\Delta}$. The natural maps*

$$\begin{gathered}
\mathsf{Hom}_{K_{\infty,\Delta}}(X_{1,\mathsf{f}}, X_{2,\mathsf{f}}) \to \mathsf{Hom}_{L_\Delta}(X_1, X_2)_{\mathsf{f}} \\
\mathsf{Hom}_{\mathbf{Rep}_{K_{\infty,\Delta}}(\Gamma_{K,\Delta})}(X_{1,\mathsf{f}}, X_{2,\mathsf{f}}) \to \mathsf{Hom}_{\mathbf{Rep}_{L_\Delta}(\Gamma_{K,\Delta})}(X_1, X_2) \\
\mathsf{Ext}^1_{\mathbf{Rep}_{K_{\infty,\Delta}}(\Gamma_{K,\Delta})}(X_{1,\mathsf{f}}, X_{2,\mathsf{f}}) \to \mathsf{Ext}^1_{\mathbf{Rep}_{L_\Delta}(\Gamma_{K,\Delta})}(X_1, X_2)
\end{gathered}$$

are bijective.

Proof. By Corollary 3.24, we have $L_\Delta \otimes_{K_{\Delta,\infty}} X_{1,\mathsf{f}} \xrightarrow{\sim} X_1$ and $L_\Delta \otimes_{K_{\infty,\Delta}} X_{2,\mathsf{f}} \xrightarrow{\sim} X_2$, so that the map

$$L_\Delta \otimes_{K_{\infty,\Delta}} \mathsf{Hom}_{K_{\infty,\Delta}}(X_{1,\mathsf{f}}, X_{2,\mathsf{f}}) \to \mathsf{Hom}_{L_\Delta}(X_1, X_2)$$

is an isomorphism of L_Δ-representations of $\Gamma_{K,\Delta}$ (since $X_{1,\mathsf{f}}$ and $X_{2,\mathsf{f}}$ are free of finite rank over $K_{\infty,\Delta}$). If we pick bases of $X_{1,\mathsf{f}}$ and $X_{2,\mathsf{f}}$ over $K_{\infty,\Delta}$, the same proof as in the previous corollary shows that there is a natural isomorphism

$$\mathsf{Hom}_{K_{\infty,\Delta}}(X_{1,\mathsf{f}}, X_{2,\mathsf{f}}) \xrightarrow{\sim} \mathsf{Hom}_{L_\Delta}(X_1, X_2)_{\mathsf{f}}$$

of $K_{\infty,\Delta}$-representations of $\Gamma_{K,\Delta}$. Taking invariants under $\Gamma_{K,\Delta}$ gives the K_Δ-linear isomorphism

$$\mathsf{Hom}_{\mathbf{Rep}_{K_{\infty,\Delta}}(\Gamma_{K,\Delta})}(X_{1,\mathsf{f}}, X_{2,\mathsf{f}}) \xrightarrow{\sim} \mathsf{Hom}_{\mathbf{Rep}_{L_\Delta}(\Gamma_{K,\Delta})}(X_1, X_2).$$

As $\mathsf{Ext}^1_{\mathbf{Rep}_{K_{\infty,\Delta}}(\Gamma_{K,\Delta})}(X_{1,\mathsf{f}}, X_{2,\mathsf{f}}) = \mathsf{H}^1(\Gamma_{K,\Delta}, X_{\mathsf{f}})$ and $\mathsf{Ext}^1_{\mathbf{Rep}_{L_\Delta}(\Gamma_{K,\Delta})}(X_1, X_2) = \mathsf{H}^1(\Gamma_{K,\Delta}, X)$ where we have put $X = \mathsf{Hom}_{L_\Delta}(X_1, X_2) \in \mathbf{Rep}_{L_\Delta}(\Gamma_{K,\Delta})$, the last statement reduces to showing that the natural map

$$\mathsf{H}^1(\Gamma_{K,\Delta}, X_{\mathsf{f}}) \to \mathsf{H}^1(\Gamma_{K,\Delta}, X)$$

is bijective. Let $c\colon \Gamma_{K,\Delta} \to X_{\mathsf{f}}$ be a cocycle whose image in $\mathsf{H}^1(\Gamma_{K,\Delta}, X)$ is trivial: there exists $x \in X$ such that $(\forall g \in \Gamma_{K,\Delta})\, c(g) = g(x) - x$. Fix $\mathfrak{B}$ a $K_{\infty,\Delta}$-basis of X_{f}. The action of $\Gamma_{K,\Delta}$ on X_{f} and X is given, in the basis $\mathfrak{B}$, by a continuous cocycle $U\colon \Gamma_{K,\Delta} \to \mathsf{GL}_d(K_{\infty,\Delta})$ (where d is the rank of X over L_Δ). Let $u_g \in K_{\infty,\Delta}^d$ (resp. $v \in L_\Delta^d$) the column vector whose coefficients are the coordinnates of $c(g)$ (resp. x) in the basis $\mathfrak{B}$: we have $u_g = U_g g(v) - v$ *i.e.* $g(v) = U_g^{-1}(u_g + v)$ for all $g \in \Gamma_{K,\Delta}$. Taking $m \in \mathbf{N}$ large enough, we can assume that U has values in $\mathsf{GL}_d(K_{m,\Delta})$ and $u_g \in K_{m,\Delta}$ for all $g \in \Gamma_{K,\Delta}$. Let $\mathcal{E}$ be the sub-$K_{m,\Delta}$-module of L_Δ generated by 1 and the entries of v: it is of finite type and stable under the action of $\Gamma_{K,\Delta}$. By Proposition 3.22, we have $\mathcal{E} \subset K_{\infty,\Delta}$, so that $x \in X_{\mathsf{f}}$, which means that the class of $\mathsf{H}^1(\Gamma_{K,\Delta}, X_{\mathsf{f}})$ is trivial. This shows the injectivity of the map. To prove the surjectivity, start from a continuous cocycle $c\colon \Gamma_{K,\Delta} \to X$. It defines an extension $\widetilde{X}$ of L_Δ par X. As L_Δ-modules, we have $\widetilde{X} = X \oplus L_\Delta$, the action of $g \in \Gamma_{K,\Delta}$ is given by $g(x,\lambda) = (g(x) + c(g)g(\lambda), g(\lambda))$. By Corollary 3.24, the $K_{\infty,\Delta}$-module $\widetilde{X}_{\mathsf{f}}$ is free of rank $d+1$ and the map $L_\Delta \otimes_{K_{\infty,\Delta}} \widetilde{X}_{\mathsf{f}} \to X$ is an isomorphism. This shows that the sequence

$$0 \to X_{\mathsf{f}} \to \widetilde{X}_{\mathsf{f}} \to K_{\Delta,\infty} \to 0$$

is exact when tensored by L_Δ over $K_{\Delta,\infty}$: as L_Δ is faithfully flat over $K_{\Delta,\infty}$ by Proposition 3.26, this implies that it is exact, so that $\widetilde{X}_{\mathsf{f}}$ defines an extension of $K_{\infty,\Delta}$ by X_{f}, that corresponds to a cohomology class in $\mathsf{H}^1(\Gamma_{K,\Delta}, X_{\mathsf{f}})$ mapping to the class of c in $\mathsf{H}^1(\Gamma_{K,\Delta}, X)$. $\square$

Theorem 3.28. *The functors*

$$\begin{array}{ccc} \mathbf{Rep}^{\mathsf{f}}_{C_\Delta}(G_{K,\Delta}) \leftrightarrow \mathbf{Rep}^{\mathsf{f}}_{L_\Delta}(\Gamma_{K,\Delta}) & & \mathbf{Rep}^{\mathsf{f}}_{L_\Delta}(\Gamma_{K,\Delta}) \leftrightarrow \mathbf{Rep}^{\mathsf{f}}_{K_{\Delta,\infty}}(\Gamma_{K,\Delta}) \\ W \mapsto W^{H_{K,\Delta}} & & X \mapsto X_{\mathsf{f}} \\ C_\Delta \otimes_{L_\Delta} X \leftarrow\!\shortmid X & & L_\Delta \otimes_{K_{\Delta,\infty}} Y \leftarrow\!\shortmid Y \end{array}$$

are equivalences of categories.

Corollary 3.29. *If $W \in \mathbf{Rep}^{\mathsf{f}}_{C_\Delta}(G_{K,\Delta})$, we have $\mathsf{H}^i(H_{K,\Delta}, W) = 0$ for all $i \in \mathbf{N}_{>0}$.*

Proof. By what precedes, we have $W \simeq C_\Delta \otimes_{K_{\infty,\Delta}} Y$ where $Y = \big(W^{H_{K,\Delta}}\big)_{\mathsf{f}} \in \mathbf{Rep}^{\mathsf{f}}_{K_{\infty,\Delta}}(\Gamma_{K,\Delta})$. As Y is free of finite rank over $K_{\infty,\Delta}$, we have

$$\mathsf{H}^i(H_{K,\Delta}, W) \simeq \mathsf{H}^i(H_{K,\Delta}, C_\Delta) \otimes_{K_{\infty,\Delta}} Y = 0$$

for all $i \in \mathbf{N}_{>0}$ by Theorem 3.3. □

3.30. Generalized Sen operators

Let Y be a free $K_{\Delta,\infty}$-representation of rank d of $\Gamma_{K,\Delta}$.

Definition. The Sen operators of Y are the maps $(\varphi_\alpha)_{\alpha\in\Delta}$ given by

$$\varphi_\alpha(y) = \lim_{\substack{\gamma\in\Gamma_{K_\alpha} \\ \gamma\to\mathsf{Id}}} \frac{\gamma(y) - y}{\log(\chi(\gamma))}.$$

Note that $\varphi_\alpha \in \mathsf{End}_{K_{\Delta,\infty}}(Y)$ for all $\alpha \in \Delta$. As $\Gamma_{K,\Delta}$ is commutative (since Γ_K is), the operators φ_α commute in $\mathsf{End}_{K_{\Delta,\infty}}(Y)$. Also, each φ_α commutes with the action of $\Gamma_{K,\Delta}$ by construction.

These operators describe the infinitesimal action of $\Gamma_{K,\Delta}$ on Y. More precisely, we have

Proposition 3.31. *(cf [22, Theorem 4]). For all $y \in Y$, there is an open subgroup $\Gamma_{K,\Delta,y} \lhd \Gamma_{K,\Delta}$ such that for all $\alpha \in \Delta$ and $\gamma \in \Gamma_{K,\alpha} \cap \Gamma_{K,\Delta,y}$, we have*

$$\gamma(y) = \exp\big(\log(\chi(\gamma))\varphi_\alpha\big)y.$$

Corollary 3.32. *The set of elements in Y on which the action of $\Gamma_{K,\Delta}$ is finite (i.e. factors through a finite quotient) is $\bigcap\limits_{\alpha\in\Delta} \mathsf{Ker}(\varphi_\alpha)$.*

Proof. Let $y \in \bigcap_{\alpha\in\Delta} \mathsf{Ker}(\varphi_\alpha)$. By Proposition 3.31, there exists an open subgroup $\Gamma_{K,\Delta,y} \lhd \Gamma_{K,\Delta}$ such that for all $\alpha \in \Delta$ and $\gamma \in \Gamma_{K,\alpha} \cap \Gamma_{K,\Delta,y}$, we have $\gamma(y) = \exp\big(\log(\chi(\gamma))\varphi_\alpha\big)y = y$, so that the action of $\Gamma_{K,\Delta}$ on y factors through the finite quotient $\Gamma_{K,\Delta}/\Gamma_{K,\Delta,y}$. Conversely, if the action of $\Gamma_{K,\Delta}$ on $y \in Y$ is finite, there exists a open normal subgroup $\Gamma_y \lhd \Gamma_{K,\Delta}$ such that $\gamma(y) = y$ for all $\gamma \in \Gamma_y$. This implies in particular that $\varphi_\alpha(y) = 0$ for all $\alpha \in \Delta$. □

Proposition 3.33. *If $\underline{n} \in \mathbf{Z}^\Delta$, the K_Δ-module $Y(\underline{n})^{\Gamma_{K,\Delta}}$ is of finite type, and vanishes for all but finitely many values of $\underline{n}$. Moreover, the map $K_{\Delta,\infty} \otimes_{K_\Delta} Y^{\Gamma_{K,\Delta}} \to Y$ is injective, and its image is precisely the set of elements in Y on which the action of $\Gamma_{K,\Delta}$ is finite.*

Proof. There exists $m_0 \in \mathbf{N}$ and a basis $\mathfrak{B} = (e_1, \ldots, e_d)$ of Y over $K_{\Delta,\infty}$ such that the cocycle describing the action of $\Gamma_{K,\Delta}$ on Y in $\mathfrak{B}$ has values in $\mathsf{GL}_d(K_{m_0,\Delta})$. If $m \in \mathbf{N}_{\geq m_0}$, put $Y_m = \bigoplus_{i=1}^{d} K_{m,\Delta} e_i$. If $\Delta' \subset \Delta$ and $\alpha \in \Delta \setminus \Delta'$, we have a semi-linear action of $\Gamma_{K,\Delta}$ on $\big(K_{m,\Delta\setminus\{\alpha\}} \otimes_{F_0} K_{\infty,\{\alpha\}}\big) \otimes_{K_{m,\Delta}} Y_m$: restricting the action to $\iota_\alpha(\Gamma_K) \subset \Gamma_{K,\Delta}$ provides a K_∞-representation of Γ_K. By [22, Theorem 6] (*cf* also [16, Proposition 2.6]), the K-vector space

$$\Big(\big(K_{m,\Delta\setminus\{\alpha\}} \otimes_{F_0} K_{\infty,\{\alpha\}}\big) \otimes_{K_{m,\Delta}} Y_m(\underline{n})\Big)^{\iota_\alpha(\Gamma_K)}$$

is finite dimensional, and vanishes for all but finitely many values of n_α (the other components of $\underline{n}$ being fixed). Making m larger if necessary, we thus may assume that it lies in $Y_m(\underline{n})$. Then

$$\Big(\big(K_{m,\Delta'} \otimes_{F_0} K_{\infty,\Delta\setminus\Delta'}\big) \otimes_{K_{m,\Delta}} Y_m(\underline{n})\Big)^{\iota_\alpha(\Gamma_K)}$$

is equal to

$$\big(K_{\Delta'\cup\{\alpha\}} \otimes_{F_0} K_{\infty,\Delta\setminus(\Delta'\cup\{\alpha\})}\big) \bigotimes_{K_{\Delta'\cup\{\alpha\}} \otimes_{F_0} K_{m,\Delta\setminus\Delta'\cup\{\alpha\}}} \Big(\big(K_{m,\Delta\setminus\{\alpha\}} \otimes_{F_0} K_{\infty,\{\alpha\}}\big) \otimes_{K_{m,\Delta}} Y_m(\underline{n})\Big)^{\iota_\alpha(\Gamma_K)}$$

inside

$$\big(K_{m,\Delta'\cup\{\alpha\}} \otimes_{F_0} K_{\infty,\Delta\setminus(\Delta'\cup\{\alpha\})}\big) \otimes_{K_{m,\Delta}} Y_m(\underline{n}).$$

A finite induction thus proves that $Y(\underline{n})^{\Gamma_{K,\Delta}}$ lies in $Y_m(\underline{n})$ if $m \gg 0$ (in particular it is of finite rank over K_Δ), and vanishes for all but finitely many values of $\underline{n}$.

Put $Y'_m = \bigcap\limits_{\alpha\in\Delta} \mathsf{Ker}(\varphi_{\alpha|Y_m}) \subset Y_m$. This is a $K_{m,\Delta}$-module of finite type endowed with a *discrete* semi-linear action of $\Gamma_{K,\Delta}$. By flatness of $K_{m,\Delta}$ over $K_{m_0,\Delta}$, we have $Y'_m \simeq K_{m,\Delta} \otimes_{K_{m_0,\Delta}} Y'_{m_0}$. Take a K-basis of Y'_{m_0}: for $m \gg 0$, it is fixed by $\Gamma_{K_m,\Delta}$, so that the action of $\Gamma_{K,\Delta}$ on Y'_m factors through $\Gamma_{K,\Delta}/\Gamma_{K_m,\Delta} \simeq \mathsf{Gal}(K_m/K)^\Delta$. By Galois descent, the natural map $K_{m,\Delta} \otimes_{K_\Delta} Y_m'^{\Gamma_{K,\Delta}} \to Y'_m$ is an isomorphism. Tensoring with $K_{\Delta,\infty}$ thus shows that

$$K_{\Delta,\infty} \otimes_{K_\Delta} Y_m'^{\Gamma_{K,\Delta}} \to Y' := \bigcap_{\alpha\in\Delta} \mathsf{Ker}(\varphi_\alpha)$$

is an isomorphism when $m \gg 0$, implying that $Y_m'^{\Gamma_{K,\Delta}} = Y'^{\Gamma_{K,\Delta}}$ when $m \gg 0$. As $Y^{\Gamma_{K,\Delta}} \subset Y' \subset Y$, we have $Y^{\Gamma_{K,\Delta}} = Y'^{,\Gamma_{K,\Delta}}$, hence an isomorphism

$$K_{\Delta,\infty} \otimes_{K_\Delta} Y^{\Gamma_{K,\Delta}} \xrightarrow{\sim} Y' \subset Y.$$

□

The previous proposition can be further refined in order to include all *Hodge-Tate* weights:

Proposition 3.34. *The natural map*

$$\bigoplus_{\underline{n}\in\mathbf{Z}^\Delta} K_{\Delta,\infty}(-\underline{n}) \otimes_{K_\Delta} \big(Y(\underline{n})\big)^{\Gamma_{K,\Delta}} \to Y$$

is injective.

Proof. Keep notations of the previous proof. Let $N \in \mathbf{N}$ and $E_N = \{-N,\dots,N\} \subset \mathbf{Z}$. Take N large enough so that $\big(Y(\underline{n})\big)^{\Gamma_{K,\Delta}}$ vanishes when $\underline{n} \notin E_N^\Delta$ (*cf* Proposition 3.33). If $\Delta' \subset \Delta$ and $\underline{n} = (n_\alpha)_{\alpha\in\Delta'} \in \mathbf{Z}^{\Delta'}$, we denote $(\underline{n}, \underline{0}_{\Delta\setminus\Delta'}) \in \mathbf{Z}^\Delta$ the element whose component of index α is n_α if $\alpha \in \Delta'$ and 0 otherwise. Assume $\alpha \in \Delta \setminus \Delta'$. We can see $\big(Y_m(\underline{n}, \underline{0}_{\Delta\setminus\Delta'})\big)^{\Gamma_{K,\Delta'}}$ (where we identify $\Gamma_{K,\Delta'}$ with $\Gamma_{K,\Delta'} \times \{\mathsf{Id}\}_{\Delta\setminus\Delta'} \subset \Gamma_{K,\Delta}$) as a $K_{\Delta'} \otimes_{F_0} K_{m,\Delta\setminus\Delta'}$-module endowed with a semi-linear action of $\Gamma_{K,\Delta\setminus\Delta'}$. We can view it as a K_m-vector space (via the map $i_\alpha\colon K_m \to K_{\Delta'} \otimes_{F_0} K_{m,\Delta\setminus\Delta'}$ given by $x \mapsto 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$, where x is the factor of index α) endowed with a semi-linear action of Γ_K (via $\iota_\alpha\colon \Gamma_K \to \Gamma_{K,\Delta\setminus\Delta'}$). By [11, Lemma

2.3.1], the map

$$\bigoplus_{q=-N}^{N} K_m(-q) \otimes_{K_{i_\alpha}} \left(\left(Y_m(\underline{n}, \underline{0}_{\Delta\setminus\Delta'})\right)^{\Gamma_{K,\Delta'}}(q)\right)^{\iota_\alpha(\Gamma_K)} \to \left(Y_m(\underline{n}, \underline{0}_{\Delta\setminus\Delta'})\right)^{\Gamma_{K,\Delta'}}$$

is injective. For all $q \in \mathbf{Z}$, let $(\underline{n}, q, \underline{0}_{\Delta\setminus(\Delta'\cup\{\alpha\})}) \in \mathbf{Z}^{\Delta}$ be the element whose components are all equal to those of $(\underline{n}, \underline{0}_{\Delta\setminus\Delta'})$, except that of index α which is equal to q. As

$$\left(Y_m(\underline{n}, q, \underline{0}_{\Delta\setminus(\Delta'\cup\{\alpha\})})\right)^{\Gamma_{K,\Delta'\cup\{\alpha\}}} = \left(\left(Y_m(\underline{n}, \underline{0}_{\Delta\setminus\Delta'})\right)^{\Gamma_{K,\Delta'}}(q)\right)^{\iota_\alpha(\Gamma_K)},$$

if we tensor with $K_{m,\Delta'}(\underline{n}) \otimes_{F_0} K_{\Delta\setminus\Delta'}$ over K_Δ and sum over all the $\underline{n} \in E_N^{\Delta'}$, we deduce that the map

$$\bigoplus_{\underline{\ell}\in E_N^{\Delta'\cup\{\alpha\}}} K_{m,\Delta'\cup\{\alpha\}}(-\underline{\ell}) \otimes_{K_{\Delta'\cup\{\alpha\}}} \left(Y_m(\underline{\ell}, \underline{0}_{\Delta\setminus(\Delta'\cup\{\alpha\})})\right)^{\Gamma_{K,\Delta'\cup\{\alpha\}}} \to \bigoplus_{\underline{n}\in E_N^{\Delta'}} \left(Y_m(\underline{n}, \underline{0}_{\Delta\setminus\Delta'})\right)^{\Gamma_{K,\Delta'}}$$

is injective. The composition of these maps for growing Δ' gives the natural map

$$\bigoplus_{\underline{n}\in E_N^{\Delta}} K_{m,\Delta}(-\underline{n}) \otimes_{K_\Delta} \left(Y_m(\underline{n})\right)^{\Gamma_{K,\Delta}} \to Y_m$$

which is thus injective. The inductive limit (as $m \to \infty$) of these is the natural map

$$\bigoplus_{\underline{n}\in \mathbf{Z}^{\Delta}} K_{\Delta,\infty}(-\underline{n}) \otimes_{K_\Delta} \left(Y(\underline{n})\right)^{\Gamma_{K,\Delta}} = \bigoplus_{\underline{n}\in E_N^{\Delta}} K_{\Delta,\infty}(-\underline{n}) \otimes_{K_\Delta} \left(Y(\underline{n})\right)^{\Gamma_{K,\Delta}} \to Y$$

which is injective as well. □

Corollary 3.35. *If $W \in \mathbf{Rep}^{\mathsf{f}}_{C_\Delta}(G_{K,\Delta})$, there are finitely many $\underline{n} \in \mathbf{Z}^{\Delta}$ such that $\left(W(\underline{n})\right)^{G_{K,\Delta}} \neq \{0\}$, and the natural map*

$$\alpha_{\mathrm{HT},0}(W)\colon \bigoplus_{\underline{n}\in\mathbf{Z}^{\Delta}} C_\Delta(-\underline{n}) \otimes_{K_\Delta} \left(W(\underline{n})\right)^{G_{K,\Delta}} \to W$$

is injective.

Notation. If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta})$, we put $\mathsf{D}_{\mathsf{Sen}}(V) = \big((C_\Delta \otimes_{\mathbf{Q}_p} V)^{H_{K,\Delta}}\big)_{\mathsf{f}} \in \mathbf{Rep}^{\mathsf{f}}_{K_{\Delta,\infty}}(\Gamma_{K,\Delta})$. By what precedes, the infinitesimal action of $\Gamma_{K,\Delta}$ on $\mathsf{D}_{\mathsf{Sen}}(V)$ endows it with $K_{\Delta,\infty}$-linear operators φ_α for $\alpha \in \Delta$. Similarly, we put $\mathsf{D}_{C_\Delta}(V) = (C_\Delta \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}}$: this is a K_Δ-module.

Corollary 3.36. *If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta})$ and $\underline{n} \in \mathbf{Z}^\Delta$, the K_Δ-module $\mathsf{D}_{C_\Delta}(V(\underline{n}))$ is of finite type and vanishes for all but finitely many $\underline{n}$. The natural map $K_{\Delta,\infty} \otimes_{K_\Delta} \mathsf{D}_{C_\Delta}(V) \to \mathsf{D}_{\mathsf{Sen}}(V)$ is injective, with image $\bigcap\limits_{\alpha\in\Delta} \mathsf{Ker}(\varphi_\alpha)$. Moreover, the natural map*

$$\alpha_{\mathrm{HT},0}(V)\colon \bigoplus_{\underline{n}\in\mathbf{Z}^\Delta} C_\Delta(-\underline{n}) \otimes_{K_\Delta} \mathsf{D}_{C_\Delta}(V(\underline{n})) \to C_\Delta \otimes_{\mathbf{Q}_p} V$$

is injective.

4. Multivariable period rings, de Rham and Hodge-Tate representations

4.1. Construction and first properties of $\mathsf{B}_{\mathrm{dR},\Delta}$

Let $C^\flat = \varprojlim\limits_{x\mapsto x^p} C$ be the tilt of C: this is an algebraically closed complete valued field of characteristic p, endowed with a continuous action of G_K. Denote by $\mathcal{O}_{C^\flat}$ its ring of integers. Recall there is a surjective map

$$\theta\colon \mathsf{W}\left(\mathcal{O}_{C^\flat}\right) \to \mathcal{O}_C$$

whose kernel is principal, generated by $\xi = p - [\widetilde{p}]$ where $\widetilde{p} = (p, p^{1/p}, \ldots) \in \mathcal{O}_{C^\flat}$. It extends into a surjective ring homomorphism $\theta\colon \mathsf{W}\left(\mathcal{O}_{C^\flat}\right)[p^{-1}] \to C$. We denote by $\mathsf{B}^+_{\mathrm{dR}}$ the completion of $\mathsf{W}\left(\mathcal{O}_{C^\flat}\right)[p^{-1}]$ with respect to the $\mathsf{Ker}(\theta) = (\xi)$-adic topology. The map θ extends into a surjective ring homomorphism $\theta\colon \mathsf{B}^+_{\mathrm{dR}} \to C$. The ring $\mathsf{B}^+_{\mathrm{dR}}$ is a DVR with uniformizer ξ and residue field C. Another unifomizer is given by

$$t = \log[\varepsilon] = -\sum_{n=1}^{\infty} \frac{(1-[\varepsilon])^n}{n}$$

where $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_{C^\flat}$ is a compatible sequence of primitive p^n-th roots of unity. The natural map $k \to \mathcal{O}_{C^\flat}$ gives rise to a ring homomorphisms $\mathsf{W}(k)[p^{-1}] \to \mathsf{W}\left(\mathcal{O}_{C^\flat}\right)[p^{-1}] \to \mathsf{B}^+_{\mathrm{dR}}$. It extends into a field extension $\overline{K} \to \mathsf{B}^+_{\mathrm{dR}}$.

Notation. • We have $\mathcal{O}_{C_\Delta}/(p) \simeq (\mathcal{O}_C/(p))^{\otimes\Delta}$ (the tensor product is taken over k). As the Frobenius map on $\mathcal{O}_C/(p)$ is surjective, the same holds on $\mathcal{O}_{C_\Delta}/(p)$. We can thus define the *tilt* of $\mathcal{O}_{C_\Delta}$ as

$$\mathcal{O}^\flat_{C_\Delta} = \varprojlim_{x\mapsto x^p} \mathcal{O}_{C_\Delta} = \Big\{(x^{(n)})_{n\in\mathbf{N}} \in \mathcal{O}_{C_\Delta}^{\mathbf{N}}\,;\,(\forall n\in\mathbf{N})\,(x^{(n+1)})^p = x^{(n)}\Big\}.$$

This is a perfect k-algebra (the map $k \to \mathcal{O}^\flat_{C_\Delta}$ being given by $x \mapsto ([x],[x^{1/p}],[x^{1/p^2}],\ldots)$), endowed with an action of $G_{K,\Delta}$, induced by its action on $\mathcal{O}_{C_\Delta}$. Moreover, the map

$$\begin{aligned}\theta_\Delta\colon \mathsf{W}(\mathcal{O}^\flat_{C_\Delta}) &\to \mathcal{O}_{C_\Delta}\\ (a_n)_{n\in\mathbf{N}} &\mapsto \sum_{n=0}^{\infty} p^n a_n^{(n)}\end{aligned}$$

is a $G_{K,\Delta}$-equivariant surjective morphism of $\mathsf{W}(k)$-algebras (*cf* [10, §5.1]). By localization, it induces a $G_{K,\Delta}$-equivariant surjective morphisms of F_0-algebras

$$\theta_\Delta\colon \mathsf{W}(\mathcal{O}^\flat_{C_\Delta})\big[\tfrac{1}{p}\big] \to C_\Delta.$$

• Put $\big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta} = \mathcal{O}_{C^\flat}\otimes_k\cdots\otimes_k\mathcal{O}_{C^\flat}$ (where the copies of $\mathcal{O}_{C^\flat}$ are indexed by Δ). If $\alpha\in\Delta$, put

$$\widetilde{p}_\alpha = 1\otimes\cdots\otimes 1\otimes\widetilde{p}\otimes 1\otimes\cdots\otimes 1$$

(where $\widetilde{p}$ is the factor of index α).

Denote by $I_{\widetilde{p}} \subset \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}$ the ideal generated by $\{\widetilde{p}_\alpha\}_{\alpha\in\Delta}$.

Lemma 4.2. *The ring $\big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}$ is $I_{\widetilde{p}}$-adically separated.*

Proof. Let $\{e_\lambda\}_{\lambda\in\Lambda}\subset\mathcal{O}_{C^\flat}$ be a subset whose image modulo $\widetilde{p}$ is a k-basis of $\mathcal{O}_{C^\flat}/(\widetilde{p})$. Then the familly $\{\widetilde{p}^i e_\lambda\}_{\substack{i\in\mathbf{N}\\ \lambda\in\Lambda}}$ is linearly independent over k, and generates a dense subspace of $\mathcal{O}_{C^\flat}$. In particular, there is an injective k-linear map $f\colon\mathcal{O}_{C^\flat}\to k^{\mathbf{N}\times\Lambda}$ such that $f\big(\widetilde{p}^n\mathcal{O}_{C^\flat}\big)\subset k^{\mathbf{N}_{\geq n}\times\Lambda}$. The tensor product of these provides an injective k-linear map $f_\Delta\colon\big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}\to k^{\mathbf{N}^\Delta\times\Lambda^\Delta}$, and $f_\Delta(I^n_{\widetilde{p}})\subset k^{E_n\times\Lambda^\Delta}$, where $E_n=\Big\{(i_\alpha)_{\alpha\in\Delta}\in\mathbf{N}^\Delta\,;\,\sum\limits_{\alpha\in\Delta} i_\alpha\geq n\Big\}$. In particular, we have $f_\Delta\Big(\bigcap\limits_{n=0}^{\infty} I^n_{\widetilde{p}}\Big)\subset\bigcap\limits_{n=0}^{\infty} k^{E_n\times\Lambda^\Delta}=\{0\}$ (since $\bigcap\limits_{n=0}^{\infty} E_n=\varnothing$), so that $\bigcap\limits_{n=0}^{\infty} I^n_{\widetilde{p}}=\{0\}$. □

Proposition 4.3. *There is a natural injective morphism of k-algebras $\big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta} \to \mathcal{O}^\flat_{C_\Delta}$, that induces an isomorphism*

$$\widetilde{\pi}_{1,\Delta}\colon \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}/I_{\widetilde{p}} \xrightarrow{\sim} \mathcal{O}_{C_\Delta}/(p).$$

Moreover, $\mathcal{O}^\flat_{C_\Delta}$ is isomorphic to the $I_{\widetilde{p}}$-adic completion of $\big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}$.

Proof. Recall that for each $m \in \mathbf{N}$, we have a surjective morphism of k-algebras

$$\begin{aligned} \pi_m\colon \mathcal{O}_{C^\flat} &\to \mathcal{O}_C/(p) \\ (x^{(n)})_{n\in\mathbf{N}} &\mapsto \overline{x^{(m)}} \end{aligned}$$

(where the k-algebra structure on $\mathcal{O}_C/(p)$ is given by $x \mapsto x^{1/p^m}$), and $\mathsf{Ker}(\pi_m) = \widetilde{p}^{p^m}\mathcal{O}_{C^\flat}$, hence an isomorphism $\widetilde{\pi}_m\colon \mathcal{O}_{C^\flat}/(\widetilde{p}^{p^m}) \xrightarrow{\sim} \mathcal{O}_C/(p)$. Taking the tensor product of these, we get an isomorphism of k-algebras $\widetilde{\pi}_{m,\Delta}\colon \big(\mathcal{O}_{C^\flat}/(\widetilde{p}^{p^m})\big)^{\otimes\Delta} \xrightarrow{\sim} \big(\mathcal{O}_C/(p)\big)^{\otimes\Delta} \simeq \mathcal{O}_{C_\Delta}/(p)$. This means that there is a natural surjective morphism of k-algebras $\pi_{m,\Delta}\colon \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta} \to \mathcal{O}_{C_\Delta}/(p)$, whose kernel is the ideal $I^{(m)}_{\widetilde{p}}$ generated by $\big\{\widetilde{p}_\alpha^{p^m}\big\}_{\alpha\in\Delta}$. The diagrams

$$\begin{tikzcd} 0 \arrow[r] & I^{(m)}_{\widetilde{p}} \arrow[r] & \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta} \arrow[r, "\pi_{m,\Delta}"] & \mathcal{O}_{C_\Delta}/(p) \arrow[r] & 0 \\ 0 \arrow[r] & I^{(m+1)}_{\widetilde{p}} \arrow[r] \arrow[u, hook] & \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta} \arrow[r, "\pi_{m+1,\Delta}"] \arrow[u, equal] & \mathcal{O}_{C_\Delta}/(p) \arrow[r] \arrow[u, "F"'] & 0 \end{tikzcd}$$

(where F is the Frobenius map) commutative. Passing to inverse limits provides the exact sequence

$$0 \to \varprojlim_m I^{(m)}_{\widetilde{p}} \to \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta} \to \mathcal{O}^\flat_{C_\Delta}.$$

As $I^{(m)}_{\widetilde{p}} \subset I^{p^m}_{\widetilde{p}}$, we have $\varprojlim_m I^{(m)} = \bigcap_{m=1}^{\infty} I^{(m)}_{\widetilde{p}} \subset \bigcap_{m=1}^{\infty} I^{p^m}_{\widetilde{p}} = \{0\}$ (*cf* Lemma 4.2), *i.e.* $\big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}$ is separated for the $I_{\widetilde{p}}$-adic topology. This provides the injective morphism $\big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta} \to \mathcal{O}^\flat_{C_\Delta}$ and the isomorphism $\widetilde{\pi}_{1,\Delta}$.

As seen above, the isomorphisms $\widetilde{\pi}_{m,\Delta}$ insert in the commutative diagrams

$$\begin{tikzcd} \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}/I_{\widetilde{p}}^{(m)} \arrow[r, "\widetilde{\pi}_{m,\Delta}"] & \mathcal{O}_{C_\Delta}/(p) \\ \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}/I_{\widetilde{p}}^{(m+1)} \arrow[r, "\widetilde{\pi}_{m+1,\Delta}"] \arrow[u, two heads] & \mathcal{O}_{C_\Delta}/(p) \arrow[u, "F"'] \end{tikzcd}$$

Passing to inverse limits gives an isomorphism $\varprojlim\limits_m \big(\mathcal{O}_{C^\flat}\big)^{\otimes\Delta}/I_{\widetilde{p}}^{(m)} \xrightarrow{\sim} \mathcal{O}_{C_\Delta}^\flat$, so that $\mathcal{O}_{C_\Delta}^\flat$ is the completion of $\mathcal{O}_{C_\Delta}$ with respect to the topology associated to the family of ideals $(I_{\widetilde{p}}^{(m)})_{m\in\mathbf{N}_{>0}}$. As $I_{\widetilde{p}}^{p^m\delta} \subset I_{\widetilde{p}}^{(m)} \subset I_{\widetilde{p}}^{p^m}$ for all $m \in \mathbf{N}_{>0}$, this topology coincides with the $I_{\widetilde{p}}$-adic topology, proving the last part of the proposition. □

Notation. For $\alpha \in \Delta$, put $\xi_\alpha = p - [\widetilde{p}_\alpha] \in \mathsf{W}(\mathcal{O}_{C_\Delta}^\flat)$. We have $\xi_\alpha \in \mathsf{Ker}(\theta_\Delta)$.

Corollary 4.4. *The ideal* $\mathsf{Ker}(\theta_\Delta)$ *is generated by* $\{\xi_\alpha\}_{\alpha\in\Delta}$.

Proof. We have $\theta_\Delta(\xi_\alpha) = 0$ for all $\alpha \in \Delta$, so that the ideal generated by $\{\xi_\alpha\}_{\alpha\in\Delta}$ lies in $\mathsf{Ker}(\theta_\Delta)$. To prove that this inclusion is an equality, it is enough to check it modulo p (because the source and the target of θ_Δ are p-adically separated and complete), *i.e.* that the kernel of the map

$$\mathcal{O}_{C_\Delta}^\flat \to \mathcal{O}_{C_\Delta}/(p)$$

(given by $x \mapsto x^{(0)}$) is the ideal generated by $I_{\widetilde{p}}$. This kernel contains $I_{\widetilde{p}}$: passing to the quotient, we get a morphism $\mathcal{O}_{C_\Delta}/I_{\widetilde{p}} \to \mathcal{O}_{C_\Delta}$, which is nothing but $\widetilde{\pi}_{1,\Delta}$. We conclude using Proposition 4.3. □

Definition. • Let $\mathsf{A}_{\mathrm{inf},\Delta}$ be the completion of $\mathsf{W}(\mathcal{O}_{C_\Delta}^\flat)$ with respect to the $\theta_\Delta^{-1}\big(p\mathcal{O}_{C_\Delta}\big)$-adic topology (*cf* [8, Définition 2.2]), where $\theta_\Delta^{-1}\big(p\mathcal{O}_{C_\Delta}\big) = \langle p, \mathsf{Ker}(\theta_\Delta)\rangle$. This is a $\mathsf{W}(k)$-algebra endowed with an action of $G_{K,\Delta}$ (because θ_Δ is equivariant). The map θ_Δ extends to a surjective $G_{K,\Delta}$-equivariant morphism of $\mathsf{W}(k)$-algebras

$$\theta_\Delta\colon\ \mathsf{A}_{\mathrm{inf},\Delta} \to \mathcal{O}_{C_\Delta}$$

(because $\mathcal{O}_{C_\Delta}$ is p-adically complete).

• Let $\mathsf{B}^+_{\mathrm{dR},\Delta}$ the completion of $\mathsf{A}_{\mathrm{inf},\Delta}\left[\frac{1}{p}\right]$ with respect to the $\mathsf{Ker}(\theta_\Delta)$-adic topology. This is a F_0-algebra endowed with an action of $G_{K,\Delta}$, and the map θ_Δ extends to a surjective θ_Δ-equivariant morphism of F_0-algebras

$$\theta_\Delta\colon\ \mathsf{B}^+_{\mathrm{dR},\Delta}\to C_\Delta.$$

By definition, we have $\mathsf{B}^+_{\mathrm{dR},\Delta}=\varprojlim_{m\geq 1}\mathsf{A}_{\mathrm{inf},\Delta}\left[\frac{1}{p}\right]/\mathsf{Ker}(\theta_\Delta)^m$. The p-adic topology on $\mathsf{A}_{\mathrm{inf},\Delta}$ induces a Banach space topolgy on each quotient $\mathsf{A}_{\mathrm{inf},\Delta}\left[\frac{1}{p}\right]/\mathsf{Ker}(\theta_\Delta)^m$ and we endow $\mathsf{B}^+_{\mathrm{dR},\Delta}$ with the inverse limit (*i.e.* product) topology, which we call the *canonical topology*. Otherwise mentioned, $\mathsf{B}^+_{\mathrm{dR},\Delta}$ is considered as a topological ring with this topology.

Remark 4.5. In contrast with $\mathsf{B}^+_{\mathrm{dR}}$, the ring $\mathsf{B}^+_{\mathrm{dR},\Delta}$ depends of k when $\delta>1$.

Proposition 4.6. *If F is a finite subextension of $\overline{K}/F_0$, the F_0-algebra structure of $\mathsf{B}^+_{\mathrm{dR},\Delta}$ extends uniquely to a F_Δ-algebra structure.*

Proof. This follows from the fact that F_Δ is a finite étale F_0-algebra. □

Notation. If $\alpha\in\Delta$, the $\mathcal{O}_{C^\flat}$-algebra structure of $\mathcal{O}^\flat_{C_\Delta}$ (induced by the map $x\mapsto 1\otimes\cdots\otimes 1\otimes x\otimes 1\otimes\cdots\otimes 1$, where x is the factor of index α) induces a $\mathsf{B}^+_{\mathrm{dR}}$-algebra structure on $\mathsf{B}^+_{\mathrm{dR},\Delta}$. All together, this provides a $\mathsf{B}^{+\otimes\Delta}_{\mathrm{dR}}$-algebra structure on $\mathsf{B}^+_{\mathrm{dR},\Delta}$. In particular, any element $b\in\mathsf{B}^+_{\mathrm{dR}}$ provides an element $b_\alpha\in\mathsf{B}^+_{\mathrm{dR},\Delta}$ (which is nothing but the image of $1\otimes\cdots\otimes 1\otimes b\otimes 1\otimes\cdots\otimes 1$). For instance, we have the elements t_α and $(p-[\widetilde{p}])_\alpha=\xi_\alpha$.

Proposition 4.7. *The ideal $\mathsf{Ker}(\theta_\Delta)\subset\mathsf{B}^+_{\mathrm{dR},\Delta}$ is generated by $\{t_\alpha\}_{\alpha\in\Delta}$.*

Proof. This follows from Corollary 4.4 and the fact that t and ξ both generate $\mathsf{Ker}(\theta)$ in $\mathsf{B}^+_{\mathrm{dR}}$, so that they differ by a unit in $\mathsf{B}^+_{\mathrm{dR}}$. □

We endow $\mathsf{B}^+_{\mathrm{dR},\Delta}$ with the filtration defined by

$$\mathsf{Fil}^i\,\mathsf{B}^+_{\mathrm{dR},\Delta}=\mathsf{Ker}(\theta_\Delta)^i$$

for all $i\in\mathbf{N}$. This filtration is decreasing and exhaustive. Let $\mathsf{gr}\,\mathsf{B}^+_{\mathrm{dR},\Delta}=\bigoplus_{i=0}^{\infty}\mathsf{gr}^i\,\mathsf{B}^+_{\mathrm{dR},\Delta}$ be the associated graded ring.

Proposition 4.8. *Let R be a ring, $x_1, \dots, x_n$ a regular sequence in R and $\widehat{R} = \varprojlim_m R/I^m$ the I-adic completion of R, where $I \subset R$ the ideal generated by $\{x_1, \dots, x_n\}$. Then the sequence $x_1, \dots, x_n$ is regular in $\widehat{R}$.*

Proof. This follows from [17, Theorem 39 (2)]. □

Lemma 4.9. *The sequence $\left(\widetilde{p}_\alpha\right)_{\alpha\in\Delta}$ is regular in $\mathcal{O}^\flat_{C_\Delta}$.*

Proof. By Proposition 4.3, $\mathcal{O}^\flat_{C_\Delta}$ is isomorphic to the $I_{\widetilde{p}}$-adic completion of $\left(\mathcal{O}_{C^\flat}\right)^{\otimes\Delta}$: Lemma 4.8 shows that it is enough to check that the sequence $\left(\widetilde{p}_\alpha\right)_{\alpha\in\Delta}$ is regular in $\left(\mathcal{O}_{C^\flat}\right)^{\otimes\Delta}$, *i.e.* that for each $\Delta' \subset \Delta$ and $\alpha \in \Delta \setminus \Delta'$, the element $\widetilde{p}_\alpha$ is regular in the quotient $\left(\mathcal{O}_{C^\flat}\right)^{\otimes\Delta} / \langle \widetilde{p}_\beta \rangle_{\beta\in\Delta'} \simeq \left(\mathcal{O}_{C^\flat}/\langle\widetilde{p}\rangle\right)^{\otimes\Delta'} \otimes_k \left(\mathcal{O}_{C^\flat}\right)^{\otimes(\Delta\setminus\Delta')}$. This follows from the regularity of $\widetilde{p}$ in the domain $\mathcal{O}_{C^\flat}$, by taking the tensor product with $\left(\mathcal{O}_{C^\flat}/\langle\widetilde{p}\rangle\right)^{\otimes\Delta'} \otimes_k \left(\mathcal{O}_{C^\flat}\right)^{\otimes(\Delta\setminus(\Delta'\cup\{\alpha\}))}$ above the field k. □

Proposition 4.10. *The sequence $\left(\xi_\alpha\right)_{\alpha\in\Delta}$ is regular in $\mathsf{B}^+_{\mathrm{dR},\Delta}$. Similarly, the sequence $(t_\alpha)_{\alpha\in\Delta}$ is regular in $\mathsf{B}^+_{\mathrm{dR},\Delta}$.*

Proof. By Lemma 4.9, the sequence $\left(\widetilde{p}_\alpha\right)_{\alpha\in\Delta}$ is regular in $\mathcal{O}^\flat_{C_\Delta}$. As p is regular in $\mathsf{W}(\mathcal{O}^\flat_{C_\Delta})$, this implies that $\left(p, \xi_\alpha\right)_{\alpha\in\Delta}$ is regular in $\mathsf{W}(\mathcal{O}^\flat_{C_\Delta})$. By Proposition 4.8, the same holds in $\mathsf{A}_{\mathrm{inf},\Delta}$. As $\mathsf{A}_{\mathrm{inf},\Delta}$ is separated with respect to the $\langle p, \xi_\alpha\rangle_{\alpha\in\Delta}$-adic topology (the same holds for $\mathsf{A}_{\mathrm{inf},\Delta} / \langle p, \xi_\alpha\rangle_{\alpha\in\Delta'}$ for any subset $\Delta' \subset \Delta$), [7, Théorème 1] implies that any permutation of the sequence $(p, \xi_\alpha)_{\alpha\in\Delta}$ is regular in $\mathsf{A}_{\mathrm{inf},\Delta}$ as well. This shows in particular that $\left(\xi_\alpha\right)_{\alpha\in\Delta}$ is regular in $\mathsf{A}_{\mathrm{inf},\Delta}$. As localization is an exact functor, this implies that $\left(\xi_\alpha\right)_{\alpha\in\Delta}$ is also regular in $\mathsf{A}_{\mathrm{inf},\Delta}\left[\frac{1}{p}\right]$. Lemma 4.8 then implies that it is also a regular sequence in $\mathsf{B}^+_{\mathrm{dR},\Delta}$. The second part of the proposition follows. □

Note that $\mathsf{gr}^0\,\mathsf{B}^+_{\mathrm{dR},\Delta} = \mathsf{B}^+_{\mathrm{dR},\Delta} / \mathsf{Ker}(\theta_\Delta) \simeq C_\Delta$. By an abuse of notation, we still denote by t_α its image in $\mathsf{gr}^1\,\mathsf{B}^+_{\mathrm{dR},\Delta}$.

Corollary 4.11. *The morphism of C_Δ-algebras $C_\Delta[X_\alpha]_{\alpha\in\Delta} \to \mathsf{gr}\,\mathsf{B}^+_{\mathrm{dR},\Delta}$ mapping X_α to t_α for all $\alpha \in \Delta$ is an isomorphism.*

Proof. This follows from Proposition 4.10 and [7, Théorème 1]. □

Corollary 4.12. *We have $\mathsf{H}^0(G_{K,\Delta}, \mathsf{B}^+_{\mathrm{dR},\Delta}) \simeq K_\Delta$.*

Proof. Let $x \in \mathsf{H}^0(G_{K,\Delta}, \mathsf{B}^+_{\mathrm{dR},\Delta})$: we have $y := \theta_\Delta(x) \in C_\Delta^{G_{K,\Delta}} = K_\Delta$ since θ_Δ is equivariant. This implies that $z := x - y \in \mathsf{H}^0(G_{K,\Delta}, \mathsf{Fil}^1 \mathsf{B}_{\mathrm{dR},\Delta})$. Assume $z \neq 0$: as the filtration is separated, there exists $i \in \mathbf{N}_{>0}$ such that $z \in \mathsf{Fil}^i \mathsf{B}^+_{\mathrm{dR},\Delta} \setminus \mathsf{Fil}^{i+1} \mathsf{B}^+_{\mathrm{dR},\Delta}$, so that the image $\overline{z}$ of z in $\mathsf{gr}^i \mathsf{B}^+_{\mathrm{dR},\Delta}$ is not zero. This implies in particular that $\mathsf{H}^0(G_{K,\Delta}, \mathsf{gr}^i \mathsf{B}^+_{\mathrm{dR},\Delta}) \neq 0$. By Corollary 4.11, we have $\mathsf{gr}^i \mathsf{B}^+_{\mathrm{dR},\Delta} \simeq \bigoplus\limits_{\substack{\underline{n} \in \mathbf{N}^\Delta \\ |\underline{n}| = i}} C_\Delta(\underline{n})$: by Theorem 3.5, this implies that $\mathsf{H}^0(G_{K,\Delta}, \mathsf{gr}^i \mathsf{B}^+_{\mathrm{dR},\Delta}) = 0$: contradiction. This shows that $z = 0$ *i.e.* $x = y \in K_\Delta$. The reverse inclusion is obvious. □

Corollary 4.13. *For all* $r \in \mathbf{N}$*, we have* $\mathsf{H}^1(H_{K,\Delta}, \mathsf{gr}^r \mathsf{B}^+_{\mathrm{dR},\Delta}) = \mathsf{H}^1(H_{K,\Delta}, \mathsf{Fil}^r \mathsf{B}^+_{\mathrm{dR},\Delta}) = \{0\}$*.*

Proof. The equalities $\mathsf{H}^1(H_{K,\Delta}, \mathsf{gr}^r \mathsf{B}^+_{\mathrm{dR},\Delta}) = \{0\}$ follow from the $G_{K,\Delta}$-modules isomorphisms $\mathsf{gr}^r \mathsf{B}^+_{\mathrm{dR},\Delta} \simeq \mathsf{Sym}^r_{C_\Delta}\big(\bigoplus\limits_{\alpha \in \Delta} C_\Delta t_\alpha\big)$ obtained by Corollary 4.11 and Theorem 3.3.
If $s > r$, we prove by induction on $s - r$ that $\mathsf{H}^1(H_{K,\Delta}, B_{r,s}) = \{0\}$, where $B_{r,s}$ is the quotient $\mathsf{Fil}^r \mathsf{B}^+_{\mathrm{dR},\Delta} / \mathsf{Fil}^s \mathsf{B}^+_{\mathrm{dR},\Delta}$. This is obvious if $s - r = 1$ since $B_{r,r+1} = \mathsf{gr}^r \mathsf{B}^+_{\mathrm{dR},\Delta}$. Assuming that $\mathsf{H}^1(H_{K,\Delta}, B_{r,s}) = \{0\}$, we have the exact sequence of $G_{K,\Delta}$-modules

$$0 \to \mathsf{gr}^s \mathsf{B}^+_{\mathrm{dR},\Delta} \to B_{r,s+1} \to B_{r,s} \to 0$$

so that $\mathsf{H}^1(H_{K,\Delta}, \mathsf{gr}^s \mathsf{B}^+_{\mathrm{dR},\Delta}) \to \mathsf{H}^1(H_{K,\Delta}, B_{r,s+1}) \to \mathsf{H}^1(H_{K,\Delta}, B_{r,s})$ is exact, hence the vanishing of $\mathsf{H}^1(H_{K,\Delta}, B_{r,s+1})$. To conclude we use the exact sequence

$$0 \to \varprojlim_s{}^{(1)} \mathsf{H}^0(\mathsf{H}_{K,\Delta}, B_{r,s}) \to \mathsf{H}^1(H_{K,\Delta}, \mathsf{Fil}^r \mathsf{B}^+_{dR,\Delta}) \to \varprojlim_s \mathsf{H}^1(H_{K,\Delta}, B_{r,s}) \to 0$$

and the fact that the sequence $\big(\mathsf{H}^0(\mathsf{H}_{K,\Delta}, B_{r,s})\big)_{s>r}$ has the Mittag-Leffler property (the transition maps are surjective, which follows from the vanishing of the cohomology of $\mathsf{gr}^s \mathsf{B}^+_{\mathrm{dR},\Delta}$ for all $s \geq r$). □

Definition. Put $t_\Delta = \prod\limits_{\alpha \in \Delta} t_\alpha \in \mathsf{B}^+_{\mathrm{dR},\Delta}$, and $\mathsf{B}_{\mathrm{dR},\Delta} = \mathsf{B}^+_{\mathrm{dR},\Delta}\left[\frac{1}{t_\Delta}\right]$. As G_K acts on t by multiplication by the cyclotomic character, the group $G_{K,\Delta}$ acts on t_Δ by multiplication by $\chi_\Delta^{\underline{1}}$, where $\underline{1}$ it the element in $\mathbf{Z}^\Delta$ whose components are all equal to 1. In particular, the action of $G_{K,\Delta}$ on $\mathsf{B}^+_{\mathrm{dR},\Delta}$ extends into an action on $\mathsf{B}_{\mathrm{dR},\Delta}$. Also, we endow $\mathsf{B}_{\mathrm{dR},\Delta} = \varinjlim_i t_\Delta^{-i} \mathsf{B}^+_{\mathrm{dR},\Delta}$ with the inductive limit topology. Finally, we endow $\mathsf{B}_{\mathrm{dR},\Delta}$ with a filtration indexed by $\mathbf{Z}$ by putting

$$\mathsf{Fil}^r \mathsf{B}_{\mathrm{dR},\Delta} = \varinjlim_{i \geq r} t_\Delta^{-i} \mathsf{Fil}^{i\delta+r} \mathsf{B}_{\mathrm{dR},\Delta}^+$$

for all $r \in \mathbf{Z}$. This defines a decreasing separated and exhaustive filtration on $\mathsf{B}_{\mathrm{dR},\Delta}$. Note that $\mathsf{Fil}^r \mathsf{B}_{\mathrm{dR},\Delta}$ is stable under the action of $G_{K,\Delta}$ for all $r \in \mathbf{Z}$.

Proposition 4.14. *We have* $\mathsf{gr}^r \mathsf{B}_{\mathrm{dR},\Delta} \simeq \bigoplus_{\substack{\underline{n}=(n_\alpha)_{\alpha\in\Delta}\in\mathbf{Z}^\Delta \\ \sum_\alpha n_\alpha = r}} C_\Delta t_\Delta^{\underline{n}}$, *where* $t_\Delta^{\underline{n}} = \prod_{\alpha\in\Delta} t_\alpha^{n_\alpha}$ *if* $\underline{n} = (n_\alpha)_{\alpha\in\Delta}$. *In particular, we have*

$$\mathsf{H}^0(G_{K,\Delta}, \mathsf{gr}^r \mathsf{B}_{\mathrm{dR},\Delta}) = \begin{cases} K_\Delta & \text{if } r = 0 \\ 0 & \text{if } r \neq 0 \end{cases}.$$

Proof. By definition we have

$$\begin{aligned} \mathsf{gr}^r \mathsf{B}_{\mathrm{dR},\Delta} &= \varinjlim_{i\geq r} t_\Delta^{-i} \mathsf{gr}^{i\delta+r} \mathsf{B}_{\mathrm{dR},\Delta}^+ \\ &\simeq \varinjlim_{i\geq r} \bigoplus_{\substack{\underline{m}=(m_\alpha)_{\alpha\in\Delta}\in\mathbf{N}^\Delta \\ \sum_\alpha m_\alpha = i\delta+r}} C_\Delta t_\Delta^{\underline{m}-i\underline{1}} = \bigoplus_{\substack{\underline{n}=(n_\alpha)_{\alpha\in\Delta}\in\mathbf{Z}^\Delta \\ \sum_\alpha n_\alpha = r}} C_\Delta t_\Delta^{\underline{n}}. \end{aligned}$$

The second part follows from Theorems 3.4 and 3.5. $\square$

Definition. Put $\mathsf{B}_{\mathrm{HT},\Delta} = \mathsf{gr}\, \mathsf{B}_{\mathrm{dR},\Delta}$. By what precedes, this is a graded C_Δ-algebra endowed with an action of $G_{K,\Delta}$, and $\mathsf{B}_{\mathrm{HT},\Delta} \simeq C_\Delta[t_\alpha, t_\alpha^{-1}]_{\alpha\in\Delta} \simeq \bigoplus_{\underline{n}\in\mathbf{Z}^\Delta} C_\Delta(\underline{n})$ (as $G_{K,\Delta}$-modules).

Corollary 4.15. *We have* $\mathsf{H}^0(G_{K,\Delta}, \mathsf{B}_{\mathrm{dR},\Delta}) = \mathsf{H}^0(G_{K,\Delta}, \mathsf{B}_{\mathrm{HT},\Delta}) = K_\Delta$.

4.16. De Rham and Hodge-Tate representations

Put $G_{F_0} = \mathsf{Gal}(\overline{K}/F_0)$. Similarly as we have seen in the proof of Proposition 3.22, the choice of an ordering $\alpha_1 < \cdots < \alpha_\delta$ of Δ provides an injective and $G_{K,\Delta}$-equivariant ring homomorphism

$$\widehat{\iota}\colon C_\Delta \to \prod_{\underline{\gamma}\in G_{F_0}^{\delta-1}} C$$

(induced by the map sending $x_1 \otimes \cdots \otimes x_\delta$ to the element whose component of index $\underline{\gamma} = (\gamma_2, \ldots, \gamma_\delta) \in G_{F_0}^{\delta-1}$ is $x_1\gamma_2(x_2\gamma_3(x_3 \cdots \gamma_\delta(x_\delta)\cdots)))$, where the action of $\underline{g} = (g_1, \ldots, g_\delta) \in G_{K,\Delta}$ on the LHS is induced by the action given by $\underline{g} \cdot (x_1 \otimes \cdots x_\delta) = g_1(x_1) \otimes \cdots \otimes g_\delta(x_\delta)$ on $\overline{K}^{\otimes\Delta}$, and that on the RHS is given by $\underline{g} \cdot (x_{\underline{\gamma}})_{\underline{\gamma}\in G_K^{\delta-1}} = \big(g_1 x_{\underline{g}\cdot\underline{\gamma}}\big)_{\underline{\gamma}}$, where $\underline{g} \cdot \underline{\gamma} = (g_1^{-1}\gamma_2 g_2, g_2^{-1}\gamma_3 g_3, \ldots, g_{\delta-1}^{-1}\gamma_\delta g_\delta)$ if $\underline{\gamma} = (\gamma_2, \ldots, \gamma_\delta)$. The tilt of this map gives an injective $G_{K,\Delta}$-equivariant ring homomorphism $\widehat{\iota}^\flat\colon C_\Delta^\flat \to \prod_{\underline{\gamma}\in G_{F_0}^{\delta-1}} C^\flat$.

The diagram

$$\begin{tikzcd}
\mathsf{W}(\mathcal{O}_{C_\Delta}^\flat) \arrow[r, hook, "\mathsf{W}(\widehat{\iota}^\flat)"] \arrow[d, "\theta_\Delta"'] & \prod_{\underline{\gamma}\in G_{F_0}^{\delta-1}} \mathsf{W}(\mathcal{O}_{C^\flat}) \arrow[r, hook] \arrow[d, "\prod_{\underline{\gamma}}\theta"] & \prod_{\underline{\gamma}\in G_{F_0}^{\delta-1}} \mathsf{B}_{\mathrm{dR}}^+ \arrow[dl] \\
\mathcal{O}_{C_\Delta} \arrow[r, "\widehat{\iota}"] & \prod_{\underline{\gamma}\in G_{F_0}^{\delta-1}} \mathcal{O}_C &
\end{tikzcd}$$

is commutative. This induces an injective and $G_{K,\Delta}$-equivariant ring homomorphism

$$\iota_{\mathrm{dR}}\colon\ \mathsf{B}_{\mathrm{dR},\Delta}^+ \to \prod_{\underline{\gamma}\in G_{F_0}^{\delta-1}} \mathsf{B}_{\mathrm{dR}}^+\,.$$

The component of index $\underline{\gamma}$ of the image of t_{α_i} by the previous map is $\chi(\gamma_2\cdots\gamma_i)t$: this shows that it extends into an injective and $G_{K,\Delta}$-equivariant map

$$\iota_{\mathrm{dR}}\colon\ \mathsf{B}_{\mathrm{dR},\Delta} \to \prod_{\underline{\gamma}\in G_{F_0}^{\delta-1}} \mathsf{B}_{\mathrm{dR}}$$

(which shows in particular that $\mathsf{B}_{\mathrm{dR},\Delta}$ is reduced).

Notation. If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta})$, we put $\mathsf{D}_{\mathrm{dR}}(V) = (\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}}$. This is a K_Δ-module which is endowed with the filtration given by $\mathsf{Fil}^r\,\mathsf{D}_{\mathrm{dR}}(V) = (\mathsf{Fil}^r\,\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}}$. By $\mathsf{B}_{\mathrm{dR},\Delta}$-linearity, the inclusion $\mathsf{D}_{\mathrm{dR}}(V) \subset \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V$ extends into a $\mathsf{B}_{\mathrm{dR},\Delta}$-linear and $G_{K,\Delta}$-equivariant map

$$\alpha_{\mathrm{dR}}(V)\colon \mathsf{B}_{\mathrm{dR},\Delta}\otimes_{K_\Delta}\mathsf{D}_{\mathrm{dR}}(V)\to \mathsf{B}_{\mathrm{dR},\Delta}\otimes_{\mathbf{Q}_p}V.$$

We define $\mathsf{D}_{\mathrm{HT}}(V)$ and $\alpha_{\mathrm{HT}}(V)$ similarly.

Proposition 4.17. *The K_Δ-modules $\mathsf{D}_{\mathrm{dR}}(V)$ and $\mathsf{D}_{\mathrm{HT}}(V)$ are of finite type, and the maps $\alpha_{\mathrm{dR}}(V)$ and $\alpha_{\mathrm{HT}}(V)$ are injective.*

Proof. • Assume that K/F_0 is Galois with group G_{K/F_0}, and put $\widetilde{\mathsf{B}}_{\mathrm{dR},\Delta}=\prod\limits_{\underline{\gamma}\in G_{F_0}^{\delta-1}}\mathsf{B}_{\mathrm{dR}}$: as recalled above, there is an injective $G_{K,\Delta}$-equivariant ring homomorphism $\iota_{\mathrm{dR}}\colon \mathsf{B}_{\mathrm{dR},\Delta}\to\widetilde{\mathsf{B}}_{\mathrm{dR},\Delta}$ (that depends on the choice of an ordering $\alpha_1<\cdots<\alpha_\delta$ on Δ). Put $\widetilde{\mathsf{D}}_{\mathrm{dR}}(V)=\big(\widetilde{\mathsf{B}}_{\mathrm{dR},\Delta}\otimes_{\mathbf{Q}_p}V\big)^{G_{K,\Delta}}\subset\prod\limits_{\underline{\gamma}\in G_{F_0}^{\delta-1}}\mathsf{B}_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}V$: the map ι_{dR} induces a K_Δ-linear injective map $\iota\colon \mathsf{D}_{\mathrm{dR}}(V)\to\widetilde{\mathsf{D}}_{\mathrm{dR}}(V)$. If $(x_{\underline{\gamma}})_{\underline{\gamma}}\in\prod\limits_{\underline{\gamma}\in G_{F_0}^{\delta-1}}\mathsf{B}_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}V$ and $\underline{g}=(g_1,\ldots,g_\delta)\in G_{K,\Delta}$, we have $\underline{g}\cdot(x_{\underline{\gamma}})_{\underline{\gamma}}=(g_1(x_{\underline{g}\cdot\underline{\gamma}}))_{\underline{\gamma}}$, thus $(x_{\underline{\gamma}})_{\underline{\gamma}}\in\widetilde{\mathsf{D}}_{\mathrm{dR}}(V)$ if and only if $g_1(x_{\underline{g}\cdot\underline{\gamma}})=x_{\underline{\gamma}}$ for all $\underline{g}\in G_{K,\Delta}$ and $\underline{\gamma}\in G_{F_0}^{\delta-1}$. If $g\in G_K$ and $\underline{\gamma}_0=(\gamma_2,\ldots,\gamma_\delta)\in G_{F_0}^{\delta-1}$, define the element $\underline{g}=(g_1,\ldots,g_\delta)\in G_{K,\Delta}$ by $g_1=g$ and $g_i=\gamma_i^{-1}g_{i-1}\gamma_i$ for all $i\in\{2,\ldots,\delta\}$ (we have indeed $\gamma_i^{-1}g_{i-1}\gamma_i\in G_K$ since G_K is normal in G_{F_0} because K/F_0 is Galois). Then we have $\underline{g}\cdot\underline{\gamma}_0=\underline{\gamma}_0$, and the component of index $\underline{\gamma}_0$ of $\underline{g}\big(x_{\underline{\gamma}}\big)_{\underline{\gamma}}$ is $g(x_{\underline{\gamma}_0})$: this shows that $x_{\underline{\gamma}_0}\in\mathsf{B}_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}V$ is fixed under G_K, *i.e.* that $x_{\underline{\gamma}_0}\in\mathsf{D}_{\mathrm{dR},\alpha_1}(V)=(\mathsf{B}_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}V)^{G_K}$ (where the action of G_K on V is via the map $\iota_{\alpha_1}\colon G_K\to G_{K,\Delta}$). This implies that $(x_{\underline{\gamma}})_{\underline{\gamma}}$ is fixed by $G_{K,\Delta}$ if and only if its components all belong to $\mathsf{D}_{\mathrm{dR},\alpha_1}(V)$ and $x_{\underline{g}\cdot\underline{\gamma}}=x_{\underline{\gamma}}$ for all $\underline{\gamma}\in G_{F_0}^{\delta-1}$ and $\underline{g}=(\mathsf{Id}_{\overline{K}},g_2,\ldots,g_\delta)\in\{\mathsf{Id}_{\overline{K}}\}\times G_K^{\delta-1}$. As $G_K^{\delta-1}$ acts transitively on those $\underline{\gamma}$ that map to a fixed $\underline{\sigma}\in G_{K/F_0}^{\delta-1}$, this shows that

$$\widetilde{\mathsf{D}}_{\mathrm{dR}}(V)=\prod_{\underline{\sigma}\in G_{K/F_0}^{\delta-1}}\mathsf{D}_{\mathrm{dR},\alpha_1}(V)\subset\prod_{\underline{\gamma}\in G_{F_0}^{\delta-1}}\mathsf{D}_{\mathrm{dR},\alpha_1}(V)$$

(where we embed the factor $\mathsf{D}_{\mathrm{dR},\alpha_1}(V)$ of index $\underline{\sigma}$ diagonally in $\prod\limits_{\substack{\underline{\gamma}\in G_{F_0}^{\delta-1}\\ \underline{\gamma}\mapsto\underline{\sigma}}}\mathsf{D}_{\mathrm{dR},\alpha_1}(V)$). As $\mathsf{D}_{\mathrm{dR},\alpha_1}(V)$ is a finite dimensional K-vector space, this shows in particular that $\widetilde{\mathsf{D}}_{\mathrm{dR}}(V)$ is a K_Δ-module of finite type.

The natural map $\mathsf{B}_{\mathrm{dR}} \otimes_K \mathsf{D}_{\mathrm{dR},\alpha_1}(V) \to \mathsf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ is injective: so are the maps

$$\Big(\prod_{\substack{\underline{\gamma}\in G_{F_0}^{\delta-1}\\ \underline{\gamma}\mapsto\underline{\sigma}}} \mathsf{B}_{\mathrm{dR}}\Big) \otimes_K \mathsf{D}_{\mathrm{dR},\alpha_1}(V) \to \Big(\prod_{\substack{\underline{\gamma}\in G_{F_0}^{\delta-1}\\ \underline{\gamma}\mapsto\underline{\sigma}}} \mathsf{B}_{\mathrm{dR}}\Big) \otimes_{\mathbf{Q}_p} V$$

for all $\underline{\sigma} \in G_{K/F_0}^{\delta-1}$, so that the map

$$\widetilde{\alpha}_{\mathrm{dR}}(V)\colon \widetilde{\mathsf{B}}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \widetilde{\mathsf{D}}_{\mathrm{dR}}(V) \to \widetilde{\mathsf{B}}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V$$

is injective (since its localizations via the projection maps $\iota_{\underline{\sigma}}\colon K_\Delta \to K$ are precisely the maps above). The diagram

$$\begin{tikzcd}
\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V) \arrow[r, "\alpha_{\mathrm{dR}}(V)"] \arrow[d, hook, "\iota_{\mathrm{dR}}\otimes 1"'] & \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V \arrow[dd, hook, "\iota_{\mathrm{dR}}\otimes 1"] \\
\widetilde{\mathsf{B}}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V) \arrow[d, hook, "1\otimes\iota"'] & \\
\widetilde{\mathsf{B}}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \widetilde{\mathsf{D}}_{\mathrm{dR}}(V) \arrow[r, hook, "\widetilde{\alpha}_{\mathrm{dR}}(V)"] & \widetilde{\mathsf{B}}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V
\end{tikzcd}$$

is commutative: this implies that $\alpha_{\mathrm{dR}}(V)$ is injective.

• If K/F_0 is not assumed to be Galois, let $K' \subset \overline{K}$ the Galois closure of K over F_0: what precedes shows that $\mathsf{D}_{\mathrm{dR},K'}(V) := (\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V)^{G_{K',\Delta}}$ is a K'_Δ-module of finite type (hence projective of finite rank) endowed with a semi-linear action of $\mathsf{Gal}(K'/K)^\Delta$, and that the natural map

$$\alpha_{\mathrm{dR},K'}(V)\colon\ \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K'_\Delta} \mathsf{D}_{\mathrm{dR},K'}(V) \to \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V$$

is injective. By Galois descent, we have $\mathsf{D}_{\mathrm{dR},K'}(V) \simeq K'_\Delta \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V)$ as $\mathsf{Gal}(K'/K)^\Delta$-modules, and $\mathsf{D}_{\mathrm{dR}}(V)$ is of finite type over K_Δ. The commutative diagram

$$\begin{tikzcd}
\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V) \arrow[dr, "\alpha_{\mathrm{dR}}(V)"] \arrow[d, equal, "\wr"'] & \\
\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K'_\Delta} \mathsf{D}_{\mathrm{dR},K'}(V) \arrow[r, "\alpha_{\mathrm{dR},K'}(V)"'] & \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V
\end{tikzcd}$$

this implies that $\alpha_{\mathrm{dR}}(V)$ is injective.

• Consider now the Hodge-Tate side. By Corollary 3.36, the K_Δ-module

$$\mathsf{D}_{\mathrm{HT}}(V) = \Big(\bigoplus_{\underline{n}\in\mathbf{Z}^\Delta} C_\Delta(\underline{n}) \otimes_{\mathbf{Q}_p} V\Big)^{G_{K,\Delta}} = \bigoplus_{\underline{n}\in\mathbf{Z}^\Delta} (C_\Delta(\underline{n}) \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}}$$

is of finite type (the sum is finite and all but finitely many factors are zero). Moreover, the natural map

$$\alpha_{\mathrm{HT},0}(V)\colon \bigoplus_{\underline{n}\in\mathbf{Z}^\Delta} C_\Delta(-\underline{n}) \otimes_{K_\Delta} (C_\Delta(\underline{n}) \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}} \to C_\Delta \otimes_{\mathbf{Q}_p} V$$

is injective and $G_{K,\Delta}$-equivariant. If we tensor with $C_\Delta(\underline{m})$ over C_Δ and sum over all $\underline{m} \in \mathbf{Z}^\Delta$, this shows that the maps in the diagram

$$\begin{tikzcd}
\bigoplus_{\underline{n},\underline{m}\in\mathbf{Z}^\Delta} C_\Delta(\underline{m}-\underline{n}) \otimes_{K_\Delta} (C_\Delta(\underline{n}) \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}} \arrow[r] \arrow[d, "\wr", equal] & \bigoplus_{\underline{m}\in\mathbf{Z}^\Delta} C_\Delta(\underline{m}) \otimes_{\mathbf{Q}_p} V \arrow[d, "\wr", equal] \\
\bigoplus_{\underline{\ell}\in\mathbf{Z}^\Delta} C_\Delta(\underline{\ell}) \otimes_{K_\Delta} \mathsf{D}_{\mathrm{HT}}(V) \arrow[r, "\alpha_{\mathrm{HT}}(V)"] & \mathsf{B}_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V
\end{tikzcd}$$

are injective. □

Definition. A p-adic representation V of $G_{K,\Delta}$ is said *de Rham* (resp. *Hodge-Tate*) when the map $\alpha_{\mathrm{dR}}(V)$ (resp. $\alpha_{\mathrm{HT}}(V)$) is bijective. We denote by $\mathbf{Rep}_{\mathrm{dR}}(G_{K,\Delta})$ (resp. $\mathbf{Rep}_{\mathrm{HT}}(G_{K,\Delta})$) the full subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta})$ whose objects are de Rham (resp. Hodge-Tate) representations.

Recall that $\mathsf{D}_{\mathrm{dR}}(V)$ is equipped with a decreasing filtration $\mathsf{Fil}^\bullet \mathsf{D}_{\mathrm{dR}}(V)$: denote by $\mathsf{gr}\,\mathsf{D}_{\mathrm{dR}}(V)$ the corresponding graded module. Note that the map $\alpha_{\mathrm{dR}}(V)$ is compatible with filtrations, where $\mathsf{Fil}^r(\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V)) = \sum_{i\in\mathbf{Z}} \mathsf{Fil}^i \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{Fil}^{r-i} \mathsf{D}_{\mathrm{dR}}(V)$ for all $i \in \mathbf{Z}$.

Proposition 4.18. *The filtration $\mathsf{Fil}^\bullet \mathsf{D}_{\mathrm{dR}}(V)$ is separated and exhaustive. There is a canonical injective map $\mathsf{gr}\,\mathsf{D}_{\mathrm{dR}}(V) \to \mathsf{D}_{\mathrm{HT}}(V)$, and $\alpha_{\mathrm{dR}}(V)$ is strictly compatible with filtrations. Moreover, if V is de Rham, then it is Hodge-Tate and the map $\mathsf{gr}\,\mathsf{D}_{\mathrm{dR}}(V) \to \mathsf{D}_{\mathrm{HT}}(V)$ is an isomorphism.*

Proof. The first assertion follows from the corresponding fact on $\mathsf{B}_{\mathrm{dR},\Delta}$ and the finiteness of $\mathsf{D}_{\mathrm{dR}}(V)$ as a K_Δ-module. If $r \in \mathbf{Z}$, the exact sequence

$$0 \to \mathsf{Fil}^{r+1}\mathsf{B}_{\mathrm{dR},\Delta} \to \mathsf{Fil}^{r}\mathsf{B}_{\mathrm{dR},\Delta} \to \mathsf{gr}^{r}\mathsf{B}_{\mathrm{dR},\Delta} \to 0$$

tensored with V induces the exact sequence

$$0 \to \mathsf{Fil}^{r+1}\mathsf{D}_{\mathrm{dR}}(V) \to \mathsf{Fil}^{r}\mathsf{D}_{\mathrm{dR}}(V) \to (\mathsf{gr}^{r}\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}}$$

i.e. an injective map $\mathsf{gr}^{r}\mathsf{D}_{\mathrm{dR}}(V) \to \bigoplus\limits_{\substack{\underline{n}=(n_\alpha)_{\alpha\in\Delta}\in\mathbf{Z}^\Delta \\ \sum_\alpha n_\alpha = r}} (C_\Delta(\underline{n}) \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}}$ (*cf* Proposition 4.14). Summing over $r \in \mathbf{Z}$ provides an injective K_Δ-linear map $i_V\colon \mathsf{gr}\,\mathsf{D}_{\mathrm{dR}}(V) \to \mathsf{D}_{\mathrm{HT}}(V)$.

As $\alpha_{\mathrm{dR}}(V)$ is compatible with filtrations, it induces a K_Δ-linear map

$$\mathsf{gr}\,\alpha_{\mathrm{dR}}(V)\colon\ \mathsf{gr}(\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V)) \to \mathsf{gr}\,\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} V.$$

We have $\mathsf{gr}(\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V)) \simeq \mathsf{gr}\,\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{gr}\,\mathsf{D}_{\mathrm{dR}}(V) \simeq \mathsf{B}_{\mathrm{HT}} \otimes_{K_\Delta} \mathsf{gr}\,\mathsf{D}_{\mathrm{dR}}(V)$ and $\mathsf{gr}\,\mathsf{B}_{\mathrm{dR},\Delta} = \mathsf{B}_{\mathrm{HT},\Delta}$ (as K_Δ is a product of fields, the proof of this isomorphism reduces to the case of tensor products of filtered vector spaces). Via these isomorphisms, we have the commutative diagram

$$\begin{tikzcd}
\mathsf{B}_{\mathrm{HT}} \otimes_{K_\Delta} \mathsf{gr}\,\mathsf{D}_{\mathrm{dR}}(V) \arrow[r, "\mathsf{gr}\,\alpha_{\mathrm{dR}}(V)"] \arrow[d, hook, "1\otimes i_V"'] & \mathsf{B}_{\mathrm{HT},\Delta} \otimes_{\mathbf{Q}_p} V \\
\mathsf{B}_{\mathrm{HT},\Delta} \otimes_{K_\Delta} \mathsf{D}_{\mathrm{HT}}(V) \arrow[ur, hook, "\alpha_{\mathrm{HT}}(V)"'] &
\end{tikzcd}$$

which implies that $\mathsf{gr}\,\alpha_{\mathrm{dR}}(V)$ is injective, so that $\alpha_{\mathrm{dR}}(V)$ is strictly compatible with filtrations.

Assume V is de Rham, so that $\alpha_{\mathrm{dR}}(V)$ is an isomorphism. Then $\mathsf{gr}\,\alpha_{\mathrm{dR}}(V)$ is an isomorphism: the preceding diagram shows that $\alpha_{\mathrm{HT}}(V)$ is surjective: it is an isomorphism by Proposition 4.17, and V is Hodge-Tate. This implies that $1 \otimes i_V$ is an isomorphism. As $\mathsf{B}_{\mathrm{HT},\Delta}$ is faithfully flat over K_Δ (because C_Δ is), this shows that i_V is an isomorphism. □

Proposition 4.19. *If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta})$ is de Rham (resp. Hodge-Tate), then $\mathsf{D}_{\mathrm{dR}}(V)$ (resp. $\mathsf{D}_{\mathrm{HT}}(V)$) is free of rank $\dim_{\mathbf{Q}_p}(V)$ over K_Δ.*

Proof. Assume V is de Rham: the map $\alpha_{\mathrm{dR}}(V)\colon \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V) \to \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V$ is an isomorphism. Use Remark 3.23 and its notations: for each $i \in I$, its localization

$$E_i \otimes \alpha_{\mathrm{dR}}(V)\colon (E_i \otimes_{K_\Delta} \mathsf{B}_{\mathrm{dR},\Delta}) \otimes_{E_i} (E_i \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V)) \\ \to (E_i \otimes_{K_\Delta} \mathsf{B}_{\mathrm{dR},\Delta}) \otimes_{\mathbf{Q}_p} V$$

is a $E_i \otimes_{K_\Delta} \mathsf{B}_{\mathrm{dR},\Delta}$-linear isomorphism. This implies that

$$\dim_{E_i}(E_i \otimes_{K_\Delta} \mathsf{D}_{\mathrm{dR}}(V)) = \dim_{\mathbf{Q}_p}(V).$$

As it holds for all $i \in I$, this shows that $\mathsf{D}_{\mathrm{dR}}(V)$ is free as a K_Δ-module. The proof of the Hodge-Tate case is the same. $\square$

Question : Is the converse true, *i.e.* is it true that if $\mathsf{D}_{\mathrm{dR}}(V)$ is free of rank $\dim_{\mathbf{Q}_p}(V)$ over K_Δ, then V is de Rham?

5. Sen theory for $\mathsf{B}^+_{\mathrm{dR},\Delta}$-representations

5.1. Almost étale descent

Put $\mathsf{L}^+_{\mathrm{dR},\Delta} = \mathsf{H}^0(H_{K,\Delta}, \mathsf{B}^+_{\mathrm{dR},\Delta})$. We have

$$\mathsf{I}^+_{\mathrm{dR},\Delta} := K_{\Delta,\infty}[\![t_\alpha]\!]_{\alpha\in\Delta} \subset \mathsf{L}^+_{\mathrm{dR},\Delta}.$$

These subrings of $\mathsf{B}^+_{\mathrm{dR},\Delta}$ are endowed with the filtration induced by the latter. In the sequel, we follow rather closely [3].

Lemma 5.2. *For all $r \in \mathbf{N}$, we have natural isomorphisms*

$$\begin{tikzcd}
\mathsf{Sym}^r_{K_{\Delta,\infty}}\Big(\bigoplus_{\alpha\in\Delta} K_{\Delta,\infty} t_\alpha\Big) \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{gr}^r\, \mathsf{I}^+_{\mathrm{dR},\Delta} \arrow[d, hook] \\
\mathsf{Sym}^r_{L_\Delta}\Big(\bigoplus_{\alpha\in\Delta} L_\Delta t_\alpha\Big) \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{gr}^r\, \mathsf{L}^+_{\mathrm{dR},\Delta} \arrow[d, hook] \\
\mathsf{Sym}^r_{C_\Delta}\Big(\bigoplus_{\alpha\in\Delta} C_\Delta t_\alpha\Big) \arrow[r, "\sim"] & \mathsf{gr}^r\, \mathsf{B}^+_{\mathrm{dR},\Delta}\,.
\end{tikzcd}$$

Moreover, the map $\mathsf{I}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^r\,\mathsf{I}^+_{\mathrm{dR},\Delta} \to \mathsf{L}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^r\,\mathsf{L}^+_{\mathrm{dR},\Delta}$ is faithfully flat.

Proof. The bottom map is an isomorphism by Corollary 4.11. This implies that the filtration on $\mathsf{I}^+_{\mathrm{dR},\Delta}$ is given by the powers of the ideal generated by $(t_\alpha)_{\alpha\in\Delta}$, showing that the top map is an isomorphism as well. To check that the middle map is also an isomorphism, we start from the exact sequence

$$0 \to \mathsf{Fil}^{r+1}\,\mathsf{B}^+_{\mathrm{dR},\Delta} \to \mathsf{Fil}^r\,\mathsf{B}^+_{\mathrm{dR},\Delta} \to \mathsf{gr}^r\,\mathsf{B}^+_{\mathrm{dR},\Delta} \to 0.$$

Taking invariants under $H_{K,\Delta}$ gives the exact sequence

$$\begin{aligned}&0 \to \mathsf{Fil}^{r+1}\,\mathsf{L}^+_{\mathrm{dR},\Delta} \to \mathsf{Fil}^r\,\mathsf{L}^+_{\mathrm{dR},\Delta}\\ &\quad\to \mathsf{H}^0(H_{K,\Delta},\mathsf{gr}^r\,\mathsf{B}^+_{\mathrm{dR},\Delta}) \to \mathsf{H}^1(H_{K,\Delta},\mathsf{Fil}^{r+1}\,\mathsf{B}^+_{\mathrm{dR},\Delta}).\end{aligned}$$

By Corollary 4.13, we have $\mathsf{H}^1(H_{K,\Delta},\mathsf{Fil}^{r+1}\,\mathsf{B}^+_{\mathrm{dR},\Delta})=\{0\}$, so that the natural map

$$\mathsf{gr}^r\,\mathsf{L}^+_{\mathrm{dR},\Delta} \to \mathsf{H}^0(H_{K,\Delta},\mathsf{gr}^r\,\mathsf{B}^+_{\mathrm{dR},\Delta}) \simeq \mathsf{Sym}^r_{L_\Delta}\Big(\bigoplus_{\alpha\in\Delta} L_\Delta t_\alpha\Big)$$

is an isomorphism. The last statement follows from Proposition 3.26. $\square$

Corollary 5.3. *Under the assumptions of Proposition 3.25, put*

$$\mathsf{L}'^+_{\mathrm{dR},\Delta} = \mathsf{H}^0(H_{K',\Delta},\mathsf{B}^+_{\mathrm{dR},\Delta}) \text{ and } \mathsf{I}'^+_{\mathrm{dR},\Delta} = K'_{\Delta,\infty}[\![t_\alpha]\!]_{\alpha\in\Delta}.$$

The natural maps $\mathsf{L}^+_{\mathrm{dR},\Delta}\to\mathsf{L}'^+_{\mathrm{dR},\Delta}$ and $\mathsf{I}^+_{\mathrm{dR},\Delta}\to\mathsf{I}'^+_{\mathrm{dR},\Delta}$ are finite étale, and Galois with group $\mathsf{Gal}(K'_\infty/K_\infty)^\Delta$ when K'_∞/K_∞ is Galois.

Proof. By Lemma 5.2 and Proposition 3.25, the natural maps $K'^{\otimes\Delta}_\infty\otimes_{K^{\otimes\Delta}_\infty}\mathsf{L}^+_{\mathrm{dR},\Delta}\to\mathsf{L}'^+_{\mathrm{dR},\Delta}$ and $K'^{\otimes\Delta}_\infty\otimes_{K^{\otimes\Delta}_\infty}\mathsf{I}^+_{\mathrm{dR},\Delta}\to\mathsf{I}'^+_{\mathrm{dR},\Delta}$ induce isomorphisms on graded rings: these are isomorphisms. The statements thus follow from Proposition 3.25. $\square$

Proposition 5.4. *(cf [3, Proposition 3.4]) The maps*

$$\varinjlim_{\substack{H\lhd H_{K,\Delta}\\ H\text{ open}}} \mathsf{H}^1(H_{K,\Delta}/H,\mathsf{GL}_d(\mathsf{B}^{+H}_{\mathrm{dR},\Delta})) \to \mathsf{H}^1(H_{K,\Delta},\mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta}))$$

$$\mathsf{H}^1(\Gamma_{K,\Delta},\mathsf{GL}_d(\mathsf{L}^+_{\mathrm{dR},\Delta})) \to \mathsf{H}^1(G_{K,\Delta},\mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta}))$$

induced by inflation maps are bijective.

Proof. Being inductive limits of inflation maps, they are injective. Let $U\colon H_{K,\Delta}\to\mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta})$ be a continuous cocycle. The composite $\theta_\Delta\circ$

$U\colon H_{K,\Delta} \to \mathsf{GL}_d(C_\Delta)$ is a continuous cocycle (by definition of the canonical topology on $\mathsf{B}^+_{\mathrm{dR},\Delta}$): by Corollary 3.11, there exists a normal open subgroup H of $H_{K,\Delta}$ (which we may and will assume of the form $H_{K',\Delta}$ for some finite Galois extension K' of K in $\overline{K}$) such that the restriction of $\theta_\Delta \circ U$ to H has a trivial cohomology class. This implies that there exists $B_0 \in \mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta})$ such that the cocycle $U_0\colon g \mapsto B_0^{-1} U_g g(B_0)$ is such that $\theta_\Delta(U_{0,g}) = \mathrm{I}_d$ for all $g \in H$.
Let $M \in \mathbf{N}$ and assume sequences $(B_m)_{0 \le m < M}$ and $(U_m)_{0 \le m < M}$ have been constructed such that:

(i) $B_m \in \mathrm{I}_d + \mathsf{M}_d(\mathsf{Fil}^m \mathsf{B}^+_{\mathrm{dR},\Delta})$ and $U_m\colon H_{K,\Delta} \to \mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta})$ is a continuous cocycle;

(ii) $U_{m,g} = B_m^{-1} U_{m-1,g} g(B_m)$ for all $g \in H_{K,\Delta}$ and $1 \le m < M$;

(iii) $U_{m,g} \in \mathsf{Id}_d + \mathsf{M}_d(\mathsf{Fil}^{m+1} \mathsf{B}^+_{\mathrm{dR},\Delta})$ for all $g \in H$ and $0 \le m < M$.

Denote by $V_{M,g}$ the image of $U_{M-1,g}$ in

$$\mathsf{M}_d(\mathsf{Fil}^M \mathsf{B}^+_{\mathrm{dR},\Delta}) / \mathsf{M}_d(\mathsf{Fil}^{M+1} \mathsf{B}^+_{\mathrm{dR},\Delta}) \simeq \mathsf{M}_d(\mathsf{gr}^M \mathsf{B}^+_{\mathrm{dR},\Delta}).$$

If $g, h \in H$, we have

$$\begin{aligned} U_{M-1,gh} - \mathrm{I}_d &= U_{M-1,g} g(U_{M-1,h}) - \mathrm{I}_d \\ &= (U_{M-1,g} - \mathrm{I}_d) g(U_{M-1,h}) + g(U_{M-1,h} - \mathrm{I}_d) \end{aligned}$$

so that

$$U_{M-1,gh} - \mathrm{I}_d \equiv U_{M-1,g} - \mathrm{I}_d + g(U_{M-1,h} - \mathrm{I}_d) \mod \mathsf{M}_d(\mathsf{Fil}^{M+1} \mathsf{B}^+_{\mathrm{dR},\Delta})$$

since

$$g(U_{M-1,h}) \equiv \mathrm{I}_d \mod \mathsf{M}_d(\mathsf{Fil}^M \mathsf{B}^+_{\mathrm{dR},\Delta}).$$

Reducing modulo $\mathsf{M}_d(\mathsf{Fil}^{M+1} \mathsf{B}^+_{\mathrm{dR},\Delta})$ gives

$$V_{M,gh} = V_{M,g} + g(V_{M,h}),$$

so that $g \mapsto V_{M,g}$ is a continuous cocycle $H = H_{K',\Delta} \to \mathsf{M}_d(\mathsf{gr}^M \mathsf{B}^+_{\mathrm{dR},\Delta})$. In particular, the entries of V_M are cocycles $H \to \mathsf{gr}^M \mathsf{B}^+_{\mathrm{dR},\Delta}$. By Corollary 4.13, these are trivial: there exists $B_M \in \mathrm{I}_d + \mathsf{M}_d(\mathsf{Fil}^M \mathsf{B}^+_{\mathrm{dR},\Delta})$ such that if we put $U_{M,g} = B_M^{-1} U_{M-1,g} g(B_M)$ for all $g \in H_{K,\Delta}$, then $U_{M,g} \in \mathsf{Id}_d + \mathsf{M}_d(\mathsf{Fil}^{M+1} \mathsf{B}^+_{\mathrm{dR},\Delta})$ for all $g \in H$.

By induction, we thus construct an infinite sequence $(B_m)_{m\in\mathbf{N}}$. Property (i) ensures that the infinite product $B = B_0B_1B_2\cdots$ converges in $\mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta})$, and condition (ii) and (iii) imply that $B^{-1}U_g g(B) = \mathrm{I}_d$ for all $g \in H$. This shows that the image of U in $\mathsf{H}^1(H, \mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta}))$ is trivial. By the inflation-restriction exact sequence

$$\begin{aligned}\{1\} &\to \mathsf{H}^1(H_{K,\Delta}/H, \mathsf{GL}_d(\mathsf{B}^{+H}_{\mathrm{dR},\Delta}))\\ &\to \mathsf{H}^1(H_{K,\Delta}, \mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta})) \to \mathsf{H}^1(H, \mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta}))\end{aligned}$$

the class of U in $\mathsf{H}^1(H_{K,\Delta}, \mathsf{GL}_d(\mathsf{B}^+_{\mathrm{dR},\Delta}))$ lies in the image of $\mathsf{H}^1(H_{K,\Delta}/H, \mathsf{GL}_d(\mathsf{B}^{+H}_{\mathrm{dR},\Delta}))$, showing the surjectivity of the first map.
The proof of the bijectivity of the second map is identical, replacing $H_{K,\Delta}$ by $G_{K,\Delta}$. Note that in that case, we can take $H = H_{K,\Delta}$ by Corollary 3.21. □

Corollary 5.5. *If W is a free $\mathsf{B}^+_{\mathrm{dR},\Delta}$-representation of rank d of $H_{K,\Delta}$, then $W^{H_{K,\Delta}}$ is a projective $\mathsf{L}^+_{\mathrm{dR},\Delta}$-module of rank d, and the natural map*

$$\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} W^{H_{K,\Delta}} \to W$$

is an isomorphism of $\mathsf{B}^+_{\mathrm{dR},\Delta}$-representations of $H_{K,\Delta}$.

Proof. By Proposition 5.4, there exists a finite Galois extension K' of K in $\overline{K}$ and a basis $\mathfrak{B}$ of W over $\mathsf{B}^+_{\mathrm{dR},\Delta}$ fixed under the action of $H_{K',\Delta}$. The $\mathsf{L}'^+_{\mathrm{dR},\Delta}$-module W' generated by $\mathfrak{B}$ coincides with $W^{H_{K',\Delta}}$(it is thus stable under the action of $H_{K,\Delta}/H_{K',\Delta} \simeq \mathsf{Gal}(K'_\infty/K_\infty)^\Delta$) and is free of rank d. Moreover, $\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}'^+_{\mathrm{dR},\Delta}} W' \to W$ is an isomorphism of $\mathsf{B}^+_{\mathrm{dR},\Delta}$-representations of $H_{K,\Delta}$. As $\mathsf{L}^+_{\mathrm{dR},\Delta} \to \mathsf{L}'^+_{\mathrm{dR},\Delta}$ is finite Galois étale with Galois group $H_{K,\Delta}/H_{K',\Delta} \simeq \mathsf{Gal}(K'_\infty/K_\infty)^\Delta$ (*cf* Corolary 5.3), we have $\mathsf{L}'^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} W'^{H_{K,\Delta}/H_{K',\Delta}} \xrightarrow{\sim} W'$ by Galois descent, and $W'^{H_{K,\Delta}/H_{K',\Delta}} = W^{H_{K,\Delta}}$ is projective of rank d over $\mathsf{L}^+_{\mathrm{dR},\Delta}$. Extending the scalars to $\mathsf{B}^+_{\mathrm{dR},\Delta}$ provides the isomorphism $\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} W^{H_{K,\Delta}} \to W$ of $\mathsf{B}^+_{\mathrm{dR},\Delta}$-representations of $H_{K,\Delta}$. □

Corollary 5.6. *(cf [16]) Let W_1, W_2 be free $\mathsf{B}^+_{\mathrm{dR},\Delta}$-representations of $H_{K,\Delta}$. Then*

$$\mathsf{Hom}_{\mathbf{Rep}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(H_{K,\Delta})}(W_1, W_2) \simeq \mathsf{Hom}_{\mathsf{L}^+_{\mathrm{dR},\Delta}}\left(W_1^{H_{K,\Delta}}, W_2^{H_{K,\Delta}}\right)$$
$$\text{and } \mathsf{Ext}^1_{\mathbf{Rep}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(H_{K,\Delta})}(W_1, W_2) = 0.$$

Proof. We have $\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} W_1^{H_{K,\Delta}} \simeq W_1$ and $\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} W_2^{H_{K,\Delta}} \simeq W_2$, thus

$$\begin{aligned}\mathsf{Hom}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(W_1, W_2) &\simeq \mathsf{Hom}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}\big(\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} W_1^{H_{K,\Delta}}, \mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} W_2^{H_{K,\Delta}}\big)\\ &\simeq \mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} \mathsf{Hom}_{\mathsf{L}^+_{\mathrm{dR},\Delta}}\big(W_1^{H_{K,\Delta}}, W_2^{H_{K,\Delta}}\big)\end{aligned}$$

as $\mathsf{B}^+_{\mathrm{dR},\Delta}$-representations of $H_{K,\Delta}$. Taking invariants under $H_{K,\Delta}$ provides an isomorphism

$$\mathsf{Hom}_{\mathbf{Rep}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(H_{K,\Delta})}(W_1, W_2) \simeq \mathsf{Hom}_{\mathsf{L}^+_{\mathrm{dR},\Delta}}\big(W_1^{H_{K,\Delta}}, W_2^{H_{K,\Delta}}\big)$$

(recall that $\mathsf{Hom}_{\mathsf{L}^+_{\mathrm{dR},\Delta}}\big(W_1^{H_{K,\Delta}}, W_2^{H_{K,\Delta}}\big)$ is projective over $\mathsf{L}^+_{\mathrm{dR},\Delta}$ since $W_1^{H_{K,\Delta}}$ and $W_2^{H_{K,\Delta}}$ are). Moreover, we have

$$\begin{aligned}&\mathsf{Ext}^1_{\mathbf{Rep}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(H_{K,\Delta})}(W_1, W_2)\\ &\quad\simeq \mathsf{Ext}^1_{\mathbf{Rep}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(H_{K,\Delta})}\big(\mathsf{B}^+_{\mathrm{dR},\Delta}, \mathsf{Hom}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(W_1, W_2)\big)\\ &\quad\simeq \mathsf{H}^1\big(H_{K,\Delta}, \mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} \mathsf{Hom}_{\mathsf{L}^+_{\mathrm{dR},\Delta}}\big(W_1^{H_{K,\Delta}}, W_2^{H_{K,\Delta}}\big)\big)\\ &\quad\simeq \mathsf{H}^1\big(H_{K,\Delta}, \mathsf{B}^+_{\mathrm{dR},\Delta}\big) \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} \mathsf{Hom}_{\mathsf{L}^+_{\mathrm{dR},\Delta}}\big(W_1^{H_{K,\Delta}}, W_2^{H_{K,\Delta}}\big)\\ &\quad= \{0\}\end{aligned}$$

since $\mathsf{Hom}_{\mathsf{L}^+_{\mathrm{dR},\Delta}}\big(W_1^{H_{K,\Delta}}, W_2^{H_{K,\Delta}}\big)$ is a projective $\mathsf{L}^+_{\mathrm{dR},\Delta}$-module and $\mathsf{H}^1\big(H_{K,\Delta}, \mathsf{B}^+_{\mathrm{dR},\Delta}\big) = \{0\}$ by Corollary 4.13. □

Definition. We say a $\mathsf{L}^+_{\mathrm{dR},\Delta}$-module X is *potentially free* if there exists a finite extension K' of K in $\overline{K}$ such that $\mathsf{L}'^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} X$ is a free $\mathsf{L}'^+_{\mathrm{dR},\Delta}$-module of finite rank. We denote by $\mathbf{Mod}^{\mathsf{pf}}(\mathsf{L}^+_{\mathrm{dR},\Delta})$ the corresponding category.

Corollary 5.7. *The functors*

$$\begin{aligned}\mathbf{Rep}^{\mathsf{f}}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(H_{K,\Delta}) &\to \mathbf{Mod}^{\mathsf{pf}}(\mathsf{L}^+_{\mathrm{dR},\Delta})\\ W &\mapsto W^{H_{K,\Delta}}\end{aligned} \quad \textit{and} \quad \begin{aligned}\mathbf{Rep}^{\mathsf{f}}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(G_{K,\Delta}) &\to \mathbf{Rep}^{\mathsf{f}}_{\mathsf{L}^+_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta})\\ W &\mapsto W^{H_{K,\Delta}}\end{aligned}$$

are equivalences of categories.

5.8. Decompletion along $K_{\Delta,\infty}/K_\Delta$

Definition. Let X be a $\mathsf{L}^+_{\mathrm{dR},\Delta}$-representation of $\Gamma_{K,\Delta}$.

(i) If X is killed by $\mathsf{Fil}^{r+1}\mathsf{L}^+_{\mathrm{dR},\Delta}$ for some $r \in \mathbf{N}$, we denote by X_{f} the union of its sub-K_Δ-modules of finite type that are stable under the action of $\Gamma_{K,\Delta}$ (this is the set of elements in X whose orbit under $\Gamma_{K,\Delta}$ generates a K_Δ-module of finite type). Note that this matches with definition of §3.6 in the case $r = 0$.

(ii) In the general case, we put $X_{\mathsf{f}} = \varprojlim_r \big(X/\mathsf{Fil}^{r+1}\mathsf{L}^+_{\mathrm{dR},\Delta}X\big)_{\mathsf{f}}$.

Note that X_{f} is a sub-$\mathsf{l}^+_{\mathrm{dR},\Delta}$-module of X and is stable by the action of $\Gamma_{K,\Delta}$.

Proposition 5.9. *Let $r \in \mathbf{N}$ and X a free $\mathsf{L}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^{r+1}\mathsf{L}^+_{\mathrm{dR},\Delta}$-representation of rank d of $\Gamma_{K,\Delta}$. Then X_{f} is free of rank d over $\mathsf{l}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^{r+1}\mathsf{l}^+_{\mathrm{dR},\Delta}$, and the natural map*

$$\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{l}^+_{\mathrm{dR},\Delta}} X_{\mathsf{f}} \to X$$

is an isomorphism of $\mathsf{L}^+_{\mathrm{dR},\Delta}$-representations of $\Gamma^+_{K,\Delta}$.

Proof. The proof is similar to that of [3, Proposition 3.17]. We use induction on r, the case $r = 0$ being Corollary 3.24: assume $r > 0$. Put $X' = \mathsf{Fil}^r\mathsf{L}^+_{\mathrm{dR},\Delta}X$: this is a free L_Δ-representation of rank $d\binom{r+\delta-1}{\delta-1}$ of $\Gamma_{K,\Delta}$; and $X'' = X/X' \simeq (\mathsf{L}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^r\mathsf{L}^+_{\mathrm{dR},\Delta}) \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} X$: this is a free $\mathsf{L}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^r\mathsf{L}^+_{\mathrm{dR},\Delta}$-representation of rank d of $\Gamma_{K,\Delta}$. By the induction hypothesis and Corollary 3.24, the natural maps

$$L_\Delta \otimes_{K_{\infty,\Delta}} X'_{\mathsf{f}} \to X'$$
$$\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{l}^+_{\mathrm{dR},\Delta}} X''_{\mathsf{f}} \to X''$$

are isomorphisms. In particular, we can find a basis $\mathfrak{B}''$ of X'' over $\mathsf{L}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^r\mathsf{L}^+_{\mathrm{dR},\Delta}$ such that the cocycle U'' giving the action of $\Gamma_{K,\Delta}$ on X'' in the basis $\mathfrak{B}''$ has values in $\mathsf{GL}_d(\mathsf{l}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^r\mathsf{l}^+_{\mathrm{dR},\Delta})$. Let $\mathfrak{B}$ be a basis of X over $\mathsf{L}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^{r+1}\mathsf{L}^+_{\mathrm{dR},\Delta}$ lifting $\mathfrak{B}''$: by construction, the cocycle U giving the action of $\Gamma_{K,\Delta}$ on X in the basis $\mathfrak{B}$ has values in $\mathsf{GL}_d\left(\mathsf{pr}_r^{-1}(\mathsf{l}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^r\mathsf{l}^+_{\mathrm{dR},\Delta})\right)$, where $\mathsf{pr}_r\colon \mathsf{L}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^{r+1}\mathsf{L}^+_{\mathrm{dR},\Delta} \to \mathsf{L}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^r\mathsf{L}^+_{\mathrm{dR},\Delta}$ is the canonical surjection. By Lemma 5.2, we have

$$\begin{aligned}\mathsf{pr}_r^{-1}(\mathsf{l}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^r\mathsf{l}^+_{\mathrm{dR},\Delta}) &= \mathsf{l}^+_{\mathrm{dR},\Delta}/\mathsf{Fil}^{r+1}\mathsf{l}^+_{\mathrm{dR},\Delta} + \mathsf{gr}^r(\mathsf{L}^+_{\mathrm{dR},\Delta})\\ &= K_{\infty,\Delta,<r}[T_\alpha]_{\alpha\in\Delta} \oplus \mathsf{gr}^r(\mathsf{L}^+_{\mathrm{dR},\Delta})\end{aligned}$$

where $K_{\infty,\Delta,<r}[T_\alpha]_{\alpha\in\Delta}$ is the sub-$K_{\infty,\Delta}$-module of elements in $K_{\infty,\Delta}[T_\alpha]_{\alpha\in\Delta}$ of total degree $< r$. For all $g \in \Gamma_{K,\Delta}$, we thus can write uniquely $U_g = \widehat{U}''_g +$

$\widetilde{U}_g$ where $\widehat{U}_g \in \mathsf{GL}_d(K_{\infty,\Delta,<r}[T_\alpha]_{\alpha\in\Delta})$ and $\widetilde{U}_g \in \mathsf{M}_d(\mathsf{gr}^r\,\mathsf{L}^+_{\mathrm{dR},\Delta})$. Note that $\widehat{U}''$ is a lift of U'', but it is not a cocycle. As $\Gamma_{K,\Delta}$ is finitely generated, we can find $n \geq n_K$ (*cf* Proposition 2.1) such that $\widehat{U}''_g \in \mathsf{GL}_d(K_{n,\Delta,<r}[T_\alpha]_{\alpha\in\Delta})$ for all $g \in \Gamma_{K,\Delta}$.

Recall that if $\alpha \in \Delta$, there is a $L_{\Delta,\Delta\setminus\{\alpha\},n}$-linear and $\Gamma_{K,\Delta}$-equivariant projector $R_{n,\alpha}\colon L_\Delta \to L_{\Delta,\Delta\setminus\{\alpha\},n}$ (*cf* notations before Theorem 3.4). They commute to each other: their composite provides a $K_{n,\Delta}$-linear projector $R_{n,\Delta}\colon L_\Delta \to K_{n,\Delta}$. We extend $R_{n,\Delta}$ to $\mathsf{gr}^r(\mathsf{L}^+_{\mathrm{dR},\Delta})$ by putting $R_{n,\Delta}(t_\alpha) = t_\alpha$ for all $\alpha \in \Delta$ (*cf* Lemma 5.2). As U is a cocycle, we have $U_{g\gamma} = U_g g(U_\gamma)$ *i.e.*

$$\begin{aligned}\widehat{U}''_{g\gamma} + \widetilde{U}_{g\gamma} &= \big(\widehat{U}''_g + \widetilde{U}_g\big) g\big(\widehat{U}''_\gamma + \widetilde{U}_\gamma\big)\\ &= \widehat{U}''_g g\big(\widehat{U}''_\gamma\big) + \widehat{U}''_g g\big(\widetilde{U}_\gamma\big) + \widetilde{U}_g g\big(\widehat{U}''_\gamma\big)\end{aligned}$$

for all $g, \gamma \in \Gamma_{K,\Delta}$, since $\widetilde{U}_g g\big(\widetilde{U}_\gamma\big) = 0$ in $\mathsf{M}_d(\mathsf{L}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{L}^+_{\mathrm{dR},\Delta})$, as $\mathsf{Fil}^{2r}\,\mathsf{L}^+_{\mathrm{dR},\Delta} \subset \mathsf{Fil}^{r+1}\,\mathsf{L}^+_{\mathrm{dR},\Delta}$. Applying $R_{n,\Delta}$, we get

$$\begin{aligned}\widehat{U}''_{g\gamma} + R_{n,\Delta}\big(\widetilde{U}_{g\gamma}\big) &= \widehat{U}''_g g\big(\widehat{U}''_\gamma\big) + \widehat{U}''_g g\big(R_{n,\Delta}\big(\widetilde{U}_\gamma\big)\big) + R_{n,\Delta}\big(\widetilde{U}_g\big) g\big(\widehat{U}''_\gamma\big)\\ &= \big(\widehat{U}''_g + R_{n,\Delta}\big(\widetilde{U}_g\big)\big) g\big(\widehat{U}''_\gamma + R_{n,\Delta}\big(\widetilde{U}_\gamma\big)\big)\end{aligned}$$

since $R_{n,\Delta}$ is $K_{n,\Delta}$-linear and $\Gamma_{K,\Delta}$-equivariant. If we put $U^{(n)}_g = \widehat{U}''_g + R_{n,\Delta}\big(\widetilde{U}_g\big)$ for all $g \in \Gamma_{K,\Delta}$, this means that $U^{(n)}\colon \Gamma_{K,\Delta} \to \mathsf{GL}_d\left(\mathsf{l}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{l}^+_{\mathrm{dR},\Delta}\right)$ is a cocycle. For $g \in \Gamma_{K,\Delta}$, put $\widetilde{U}^{(n)}_g = U_g - U^{(n)}_g \in \mathsf{M}_d\left(\mathsf{gr}^r\,\mathsf{L}^+_{\mathrm{dR},\Delta}\right)$. By the computation above (with $U^{(n)}$ and $\widetilde{U}^{(n)}$ instead of $\widehat{U}''$ and $\widetilde{U}$), we have

$$U^{(n)}_{g\gamma} + \widetilde{U}^{(n)}_{g\gamma} = U^{(n)}_g g\big(U^{(n)}_\gamma\big) + U^{(n)}_g g\big(\widetilde{U}^{(n)}_\gamma\big) + \widetilde{U}^{(n)}_g g\big(U^{(n)}_\gamma\big)$$

so that

$$\widetilde{U}^{(n)}_{g\gamma} = U^{(n)}_g g\big(\widetilde{U}^{(n)}_\gamma\big) + \widetilde{U}^{(n)}_g g\big(U^{(n)}_\gamma\big) = U_g g\big(\widetilde{U}^{(n)}_\gamma\big) + \widetilde{U}^{(n)}_g g\big(U_\gamma\big)$$

since $U^{(n)}$ is a cocycle. This means that $\widetilde{U}^{(n)}$ is a cocycle that defines an extension of $L_\Delta \otimes_{K_{\infty,\Delta}} X''_{\mathsf{f}}$ by $L_\Delta \otimes_{K_{\infty,\Delta}} X'_{\mathsf{f}}$ as L_Δ-representations of $\Gamma_{K,\Delta}$. By Corollary 3.27, this extension comes from an extension of X''_{f} by X'_{f} as $K_{\infty,\Delta}$-representations of $\Gamma_{K,\Delta}$, by extension of scalars: there exists $N \in$

$\mathsf{M}_d(\mathsf{gr}^r\,\mathsf{L}^+_{\mathrm{dR},\Delta})$ such that

$$\widetilde{U}_g^{(n)} + U_g g(N) - NU_g \in \mathsf{M}_d(\mathsf{gr}^r\,\mathsf{I}^+_{\mathrm{dR},\Delta})$$

for all $g\in\Gamma_{K,\Delta}$. Now put $B = \mathrm{I}_d + N \in \mathsf{GL}_d(\mathsf{L}^+_{\mathrm{dR},\Delta})$: for all $g\in\Gamma_{K,\Delta}$ we have

$$\begin{aligned} B^{-1}U_g g(B) &= (\mathrm{I}_d - N)\big(U_g^{(n)} + \widetilde{U}_g^{(n)}\big)(\mathrm{I}_d + g(N)) \\ &= U_g^{(n)} + \widetilde{U}_g^{(n)} - NU_g^{(n)} + U_g^{(n)} g(N) \in \mathsf{GL}_d(\mathsf{I}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta}). \end{aligned}$$

This means that if we make a change of basis from $\mathfrak{B}$ using B, we reduce to the case where the cocycle U takes values in $\mathsf{GL}_d(\mathsf{I}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta})$. Proposition 3.22 then shows that X_{f} is nothing but the $\mathsf{I}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta}$-span of the basis just constructed: in particular, it is free, and the map

$$\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} X_{\mathsf{f}} \to X$$

is an isomorphism of $\mathsf{L}^+_{\mathrm{dR},\Delta}$-representations of $\Gamma^+_{K,\Delta}$. □

Proposition 5.10. *Let X le a free $\mathsf{L}^+_{\mathrm{dR},\Delta}$-representation of rank d of $\Gamma_{K,\Delta}$. Then X_{f} is free of rank d over $\mathsf{I}^+_{\mathrm{dR},\Delta}$, and the natural map*

$$\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} X_{\mathsf{f}} \to X$$

is an isomorphism of $\mathsf{L}^+_{\mathrm{dR},\Delta}$-representations of $\Gamma^+_{K,\Delta}$. Moreover, X_{f} is the union of the sub-$\mathsf{I}^+_{\mathrm{dR},\Delta}$-modules of X that are of finite type and stable under the action of $\Gamma_{K,\Delta}$.

Proof. For $r\in\mathbf{N}$, put $X_r = (\mathsf{L}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{L}^+_{\mathrm{dR},\Delta}) \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} X$. This is a free $\mathsf{L}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{L}^+_{\mathrm{dR},\Delta}$-representation of rank d of $\Gamma_{K,\Delta}$.
By Proposition 5.9, $X_{r,\mathsf{f}}$ is a free $\mathsf{I}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta}$-representation of rank d of $\Gamma_{K,\Delta}$, and the natural map $\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} X_{r,\mathsf{f}} \to X_r$ is an isomorphism of $\mathsf{L}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{L}^+_{\mathrm{dR},\Delta}$-representations of $\Gamma_{K,\Delta}$. The maps $X_{r+1}\to X_r$ are surjective: so are the maps $X_{r+1,\mathsf{f}}\to X_{r,\mathsf{f}}$ by faithful flatness of $\mathsf{L}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{L}^+_{\mathrm{dR},\Delta}$ over $\mathsf{I}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta}$ (*cf* Lemma 5.2). This implies that any basis of $X_{r,\mathsf{f}}$ can be lifted to a basis of $X_{r+1,\mathsf{f}}$: by induction, we can construct a sequence $(\mathfrak{B}_r)_{r\in\mathbf{N}}$ such that $\mathfrak{B}_r$ is a basis of $X_{r,\mathsf{f}}$ over $\mathsf{I}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta}$ and is the image of $\mathfrak{B}_{r+1}$ in $(\mathsf{I}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta}) \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} X_{r+1,\mathsf{f}}$ for all $r\in\mathbf{N}$. This sequence defines a basis $\mathfrak{B}$ of $X_{\mathsf{f}} = \varprojlim_r X_{r,\mathsf{f}}$ over

$\mathsf{l}^+_{\mathrm{dR},\Delta}$. By extension of scalars, $\mathfrak{B}$ is a basis of X over $\mathsf{L}^+_{\mathrm{dR},\Delta}$ as well, so that the natural map

$$\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{l}^+_{\mathrm{dR},\Delta}} X_{\mathsf{f}} \to X$$

is an isomorphism of $\mathsf{L}^+_{\mathrm{dR},\Delta}$-representations of $\Gamma_{K,\Delta}$.

Let X'_{f} be the union of the sub-$\mathsf{l}^+_{\mathrm{dR},\Delta}$-modules of X that are of finite type and stable under the action of $\Gamma_{K,\Delta}$: we have $X_{\mathsf{f}} \subset X'_{\mathsf{f}}$. The reverse inclusion is checked modulo $\mathsf{Fil}^{r+1}\,\mathsf{l}^+_{\mathrm{dR},\Delta}$ for all $r \in \mathbf{N}$. Let Y be a finite type sub-$\mathsf{l}^+_{\mathrm{dR},\Delta}$-module of X that is stable under the action of $\Gamma_{K,\Delta}$, and $Y_r = (\mathsf{l}^+_{\mathrm{dR},\Delta}/\,\mathsf{Fil}^{r+1}\,\mathsf{l}^+_{\mathrm{dR},\Delta}) \otimes_{\mathsf{L}^+_{\mathrm{dR},\Delta}} Y \subset X_r$ (*cf* Lemma 5.2). Let $y_1, \ldots, y_s$ be a generating family of Y_r over $\mathsf{l}^+_{\mathrm{dR},\Delta}$ and $g_1, \ldots, g_u$ a finite set of topological generators of $\Gamma_{K,\Delta}$. There exists $n \in \mathbf{N}$ and coefficients $(c_{i,j}^{(a)})_{\substack{1\le i,j\le s\\ 1\le a\le u}}$ in $K_{n,\Delta}$ such that $g_a(y_i) = \sum\limits_{j=1}^{s} c_{i,j}^{(a)} y_j$. This implies that for all $m \ge n$, the sub-$K_{m,\Delta}$-module $Y_{r,m}$ of Y_r generated by $y_1, \ldots, y_s$ is stable under the action of $\Gamma_{K,\Delta}$: we have $Y_{r,m} \subset X_{r,\mathsf{f}}$. As $Y_r = \bigcup\limits_{m\ge n} Y_{r,m}$, we thus have $Y_r \subset X_{r,\mathsf{f}}$. □

Theorem 5.11. *The functor*

$$\begin{aligned}\mathbf{Rep}^{\mathsf{f}}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(G_{K,\Delta}) &\to \mathbf{Rep}^{\mathsf{f}}_{\mathsf{l}^+_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta})\\ W &\mapsto \big(W^{H_{K,\Delta}}\big)_{\mathsf{f}}\end{aligned}$$

is an equivalence of categories.

Proof. This follows from Corollary 5.7 and Proposition 5.10. □

Notation. Put $\mathsf{L}_{\mathrm{dR},\Delta} = \mathsf{L}^+_{\mathrm{dR},\Delta}\big[\frac{1}{t_\Delta}\big]$ and $\mathsf{l}_{\mathrm{dR},\Delta} = \mathsf{l}^+_{\mathrm{dR},\Delta}\big[\frac{1}{t_\Delta}\big] = K_{\Delta,\infty}[\![t_\alpha]\!]\big[\frac{1}{t_\alpha}\big]_{\alpha\in\Delta}$. We have $\mathsf{l}_{\mathrm{dR},\Delta} \subset \mathsf{L}_{\mathrm{dR},\Delta} = \mathsf{H}^0(H_{K,\Delta}, \mathsf{B}_{\mathrm{dR},\Delta})$.

Definition. (1) A *lattice* of a free $\mathsf{B}_{\mathrm{dR},\Delta}$-module (resp. $\mathsf{l}_{\mathrm{dR},\Delta}$-module) M of finite rank, is a sub-$\mathsf{B}^+_{\mathrm{dR},\Delta}$-module (resp. sub-$\mathsf{l}^+_{\mathrm{dR},\Delta}$-module) generated by a basis of M.

(2) The category of *regular representations* $\mathbf{Rep}^{\mathsf{reg}}_{\mathsf{B}_{\mathrm{dR},\Delta}}(G_{K,\Delta})$ (resp. $\mathbf{Rep}^{\mathsf{reg}}_{\mathsf{l}_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta})$) is the isogeny category of $\mathbf{Rep}^{\mathsf{f}}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(G_{K,\Delta})$ (resp. $\mathbf{Rep}^{\mathsf{f}}_{\mathsf{l}^+_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta})$), *i.e.* the category whose objects are free $\mathsf{B}_{\mathrm{dR},\Delta}$-modules (resp. $\mathsf{l}_{\mathrm{dR},\Delta}$-modules) of finite rank endowed with a semi-linear action of $G_{K,\Delta}$ (resp. $\Gamma_{K,\Delta}$), and admitting a $G_{K,\Delta}$-stable (resp.

$\Gamma_{K,\Delta}$-stable) lattice which defines an object of $\mathbf{Rep}_{\mathsf{B}^+_{\mathrm{dR},\Delta}}(G_{K,\Delta})$ (resp. $\mathbf{Rep}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta})$).

Theorem 5.12. *The functor*

$$\mathbf{Rep}^{\mathsf{reg}}_{\mathsf{B}_{\mathrm{dR},\Delta}}(G_{K,\Delta}) \to \mathbf{Rep}^{\mathsf{reg}}_{\mathsf{I}_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta})$$

induced by that of Theorem 5.11 is an equivalence of categories.

5.13. The module with connection associated with a $\mathsf{I}^+_{\mathrm{dR},\Delta}$-representation

Notation. Let $\Omega^+ = \Omega^+_{\mathsf{I}^+_{\mathrm{dR},\Delta}/K_{\Delta,\infty}}$ (resp. $\Omega = \Omega^+_{\mathsf{I}_{\mathrm{dR},\Delta}/K_{\Delta,\infty}}$) be the $\mathsf{I}^+_{\mathrm{dR},\Delta}$-module of continuous $K_{\Delta,\infty}$-differentials of $\mathsf{I}^+_{\mathrm{dR},\Delta}$ (resp. $\mathsf{I}_{\mathrm{dR},\Delta}$) having poles of order ≤ 1 on the divisor $(t_\Delta = 0)$. This is the free $\mathsf{I}^+_{\mathrm{dR},\Delta}$-module (resp. $\mathsf{I}_{\mathrm{dR},\Delta}$-module) with basis $\left(\frac{\mathrm{d}\, t_\alpha}{t_\alpha}\right)_{\alpha\in\Delta}$.

Definition. A *module with connection* over $\mathsf{I}^+_{\mathrm{dR},\Delta}$ is a free $\mathsf{I}^+_{\mathrm{dR},\Delta}$-module Y of finite rank equipped with a $K_{\Delta,\infty}$-linear map

$$\nabla_Y \colon Y \to Y \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} \Omega^+$$

satisfying the Leibniz rule: $\nabla_Y(\lambda y) = y \otimes \mathrm{d}\, y + \lambda\nabla_Y(y)$ for all $\lambda \in \mathsf{I}^+_{\mathrm{dR},\Delta}$ and $y \in Y$. If Y_1 and Y_2 are two modules with connection over $\mathsf{I}^+_{\mathrm{dR},\Delta}$, we endow $Y_1 \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} Y_2$ (resp. $\mathsf{Hom}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}(Y_1, Y_2)$) with the connection $\nabla_{Y_1} \otimes \mathsf{Id}_{Y_2} + \mathsf{Id}_{Y_1} \otimes \nabla_{Y_2}$ (resp. $f \mapsto \nabla_{Y_2} \circ f - (f \otimes \mathsf{Id}_{\Omega^+}) \circ \nabla_{Y_1}$). A morphism between two modules with connection is an horizontal $\mathsf{I}^+_{\mathrm{dR},\Delta}$-linear map. This defines a tensor category denoted $\mathscr{R}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}$.

Similarly, one defines the category $\mathscr{R}_{\mathsf{I}_{\mathrm{dR},\Delta}}$ of free $\mathsf{I}_{\mathrm{dR},\Delta}$-modules of finite rank with connection. If $(Y, \nabla_Y) \in \mathscr{R}_{\mathsf{I}_{\mathrm{dR},\Delta}}$, the connection ∇_Y is said *regular* if there exists a lattice $\mathcal{Y}$ in Y such that $\nabla_Y(\mathcal{Y}) \subset \mathcal{Y} \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} \Omega^+$ so that $(\mathcal{Y}, \nabla_{Y|\mathcal{Y}}) \in \mathscr{R}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}$. We denote $\mathscr{R}^{\mathsf{reg}}_{\mathsf{I}_{\mathrm{dR},\Delta}}$ the full subcategory of $\mathscr{R}_{\mathsf{I}_{\mathrm{dR},\Delta}}$ made of modules with regular connection.

Notation. Let $Y \in \mathbf{Rep}^{\mathsf{f}}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta})$. If $r \in \mathbf{N}$, the quotient $Y_r := \left(\mathsf{I}^+_{\mathrm{dR},\Delta}/\operatorname{\mathsf{Fil}}^{r+1}\mathsf{I}^+_{\mathrm{dR},\Delta}\right) \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} Y$ is a free $\mathsf{I}^+_{\mathrm{dR},\Delta}/\operatorname{\mathsf{Fil}}^{r+1}\mathsf{I}^+_{\mathrm{dR},\Delta}$-module of finite rank endowed with a continuous semi-linear action of $\Gamma_{K,\Delta}$: in particular, it defines an object of $\mathbf{Rep}^{\mathsf{f}}_{K_{\Delta,\infty}}(\Gamma_{K,\Delta})$ (*cf* Lemma 5.2). The infinitesimal action of $\Gamma_{K,\Delta}$ on Y_r provides Sen operators $(\nabla_{Y_r,\alpha})_{\alpha\in\Delta}$: these are $K_{\Delta,\infty}$-linear endomorphisms of Y_r characterized by the fact that for all $y \in Y_r$, there exists an open normal subgroup $\Gamma_{K,\Delta,y} \lhd \Gamma_{K,\Delta}$ such that for all $\alpha \in \Delta$ and $\gamma \in \Gamma_{K,\alpha} \cap \Gamma_{K,\Delta,y}$, we have $\gamma(y) = \exp\big(\log(\chi(\gamma))\nabla_{Y_r,\alpha}\big)y$ (*cf* Section 3.30).

These maps are compatible as r grows: we thus obtain $K_{\Delta,\infty}$-linear endomorphisms $(\nabla_{Y,\alpha})_{\alpha\in\Delta}$ of Y such that for all $y \in Y$ and all $r \in \mathbf{N}$, there exists an open normal subgroup $\Gamma_{K,\Delta,y,r} \lhd \Gamma_{K,\Delta}$ such that for all $\alpha \in \Delta$ and $\gamma \in \Gamma_{K,\alpha} \cap \Gamma_{K,\Delta,y,r}$, we have

$$\gamma(y) \equiv \exp\big(\log(\chi(\gamma))\nabla_{Y_r,\alpha}\big)y \mod (\mathsf{Fil}^{r+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta})Y.$$

Proposition 5.14. *Let* $Y \in \mathbf{Rep}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}(\Gamma_{K\Delta})$ *and* $\alpha \in \Delta$. *Then* $\nabla_{Y,\alpha}(\gamma(y)) = \gamma(\nabla_{Y,\alpha}(y))$ *for all* $y \in Y$ *and* $\gamma \in \Gamma_{K,\Delta}$. *Moreover, if* $y \in Y$ *and* $\beta \in \Delta$, *we have*

$$\nabla_{Y,\alpha}(t_\beta y) = \begin{cases} t_\beta \nabla_{Y,\alpha}(y) & \text{if } \beta \neq \alpha \\ t_\alpha y + t_\alpha \nabla_{Y,\Delta}(y) & \text{if } \beta = \alpha \end{cases}.$$

Proof. This is checked modulo Fil^{r+1} for all $r \in \mathbf{N}$. The first statement follows from the corresponding property of Sen operators. For the second one, fix $y \in Y_r$: we have $\nabla_{Y_r,\alpha}(t_\beta y) = \lim\limits_{\substack{\gamma\in\Gamma_{K_\alpha}\\ \gamma\to\mathsf{Id}}} \frac{\gamma(t_\beta y)-t_\beta y}{\log(\chi(\gamma))}$. If $\gamma \in \Gamma_{K_\alpha}$ and $\beta \neq \alpha$, we have $\gamma(t_\beta y) = t_\beta\gamma(y)$, so that $\nabla_{Y_r,\alpha}(t_\beta y) = t_\beta \nabla_{Y_r,\alpha}(y)$: assume $\beta = \alpha$. We have $\gamma(t_\alpha y) = \chi(\gamma)t_\alpha\gamma(y)$, so that

$$\frac{\gamma(t_\alpha y) - t_\alpha y}{\log(\chi(\gamma)} = \frac{\chi(\gamma)t_\alpha\gamma(y) - t_\alpha y}{\log(\chi(\gamma)} = t_\alpha\Big(\frac{\chi(\gamma)-1}{\log(\chi(\gamma)}\gamma(y) + \frac{\gamma(y)-y}{\log(\chi(\gamma)}\Big)$$

which converges to $t_\alpha(y + \nabla_{Y_r,\alpha}(y))$ as γ converges to Id. □

Definition. Let $Y \in \mathbf{Rep}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}(\Gamma_{K\Delta})$. If $y \in Y$, we put

$$\nabla_Y(y) = \sum_{\alpha\in\Delta} \nabla_{Y,\alpha}(y) \otimes \frac{\mathrm{d}\,t_\alpha}{t_\alpha} \in Y \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} \Omega^+.$$

Proposition 5.15. *Let* $Y \in \mathbf{Rep}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}(\Gamma_{K\Delta})$. *The map* ∇_Y *is an integrable connection.*

Proof. The Leibniz rule follows from Proposition 5.14, and the integrability from the fact that $\Gamma_{K,\Delta}$ is abelian. □

We thus have a functor

$$\mathbf{Rep}^{\mathsf{f}}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta}) \to \mathscr{R}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}$$

which induces functors

$$\mathbf{Rep}^{\mathsf{reg}}_{\mathsf{I}_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta}) \to \mathscr{R}^{\mathsf{reg}}_{\mathsf{I}_{\mathrm{dR},\Delta}} \text{ and } \mathbf{Rep}^{\mathsf{reg}}_{\mathsf{B}_{\mathrm{dR},\Delta}}(G_{K,\Delta}) \to \mathscr{R}^{\mathsf{reg}}_{\mathsf{I}_{\mathrm{dR},\Delta}}.$$

Let $Y \in \mathbf{Rep}^{\mathsf{f}}_{\mathsf{I}^+_{\mathrm{dR},\Delta}}(\Gamma_{K,\Delta})$. If $y \in Y^{\Gamma_{K,\Delta}}$, we have $\nabla_{Y,\alpha}(y) = 0$ for all $\alpha \in \Delta$, hence $\nabla_Y(y) = 0$. By $K_{\Delta,\infty}$-linearity, the inclusion $Y^{\Gamma_{K,\Delta}} \subset Y^{\nabla_Y=0}$ induces a map

$$c_\nabla(Y)\colon K_{\Delta,\infty} \otimes_{K_\Delta} Y^{\Gamma_{K,\Delta}} \to Y^{\nabla_Y=0}.$$

Proposition 5.16. *The K_Δ-module $Y^{\Gamma_{K,\Delta}}$ is of finite type and the map $c_\nabla(Y)$ is an isomorphism.*

Proof. If $r \in \mathbf{N}$, put $Y_r = (\mathsf{I}^+_{\mathrm{dR},\Delta} / \operatorname{\mathsf{Fil}}^{r+1} \mathsf{I}^+_{\mathrm{dR},\Delta}) \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} Y$: this is an object of $\mathbf{Rep}^{\mathsf{f}}_{K_{\Delta,\infty}}(\Gamma_{K,\Delta})$. Put also

$$Y_r^{\nabla_Y=0} = \{y \in Y_r\,;\, (\forall \alpha \in \Delta)\, \nabla_{Y_r,\alpha}(y) = 0\}$$

(this is a an abuse of notation since ∇_Y does not make sense on Y_r). By Proposition 3.33, the natural map

$$K_{\Delta,\infty} \otimes_{K_\Delta} Y_r^{\Gamma_{K,\Delta}} \to Y_r^{\nabla_Y=0}$$

is an $K_{\Delta,\infty}$-linear isomorphism. We have $Y^{\nabla_Y=0} = \varprojlim_r Y_r^{\nabla_Y=0}$, so in particular the inverse limit of the above isomorphisms is an isomorphism

$$\varprojlim_r K_{\Delta,\infty} \otimes_{K_\Delta} Y_r^{\Gamma_{K,\Delta}} \xrightarrow{\sim} Y^{\nabla_Y=0}.$$

Similarly, we have $Y^{\Gamma_{K,\Delta}} = \varprojlim_r Y_r^{\Gamma_{K,\Delta}}$. The exact sequence

$$0 \to \operatorname{\mathsf{gr}}^r Y \to Y_{r+1} \to Y_r \to 0$$

induces the exact sequence

$$0 \to \big(\operatorname{\mathsf{gr}}^r Y\big)^{\Gamma_{K,\Delta}} \to Y_{r+1}^{\Gamma_{K,\Delta}} \to Y_r^{\Gamma_{K,\Delta}}.$$

We have $\mathsf{gr}^r Y \simeq \bigoplus\limits_{\substack{\underline{n}\in\mathbf{N}^\Delta\\ |\underline{n}|=r}} Y_1(\underline{n})$, so that $\big(\mathsf{gr}^r Y\big)^{\Gamma_{K,\Delta}} \simeq \bigoplus\limits_{\substack{\underline{n}\in\mathbf{N}^\Delta\\ |\underline{n}|=r}} \big(Y_1(\underline{n})\big)^{\Gamma_{K,\Delta}}$. By Proposition 3.33 again, we know that $\big(Y_1(\underline{n})\big)^{\Gamma_{K,\Delta}} = \{0\}$ for all but finitely many values of $\underline{n}$. This implies that $\big(\mathsf{gr}^r Y\big)^{\Gamma_{K,\Delta}} = \{0\}$, *i.e.* that the natural map $Y_{r+1}^{\Gamma_{K,\Delta}} \to Y_r^{\Gamma_{K,\Delta}}$ is injective when $r \gg 0$. As $Y_r^{\Gamma_{K,\Delta}}$ is a K_Δ-module of finite type (hence a F_0-space of finite dimension) for all $r \in \mathbf{N}$, this implies that $Y_{r+1}^{\Gamma_{K,\Delta}} \to Y_r^{\Gamma_{K,\Delta}}$ is in fact an isomorphism when $r \gg 0$. In particular, the map $Y^{\Gamma_{K,\Delta}} \to Y_r^{\Gamma_{K,\Delta}}$ is an isomorphism for $r \gg 0$ (this proves the first part of the proposition), and $\varprojlim\limits_r K_{\Delta,\infty} \otimes_{K_\Delta} Y_r^{\Gamma_{K,\Delta}} \simeq K_{\Delta,\infty} \otimes_{K_\Delta} Y^{\Gamma_{K,\Delta}}$. $\square$

5.17. Application to p-adic representations: link with multivariate (φ, Γ)-modules

Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta})$. Then $\mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V \in \mathbf{Rep}^{\mathsf{reg}}_{\mathsf{B}_{\mathrm{dR},\Delta}}(G_{K,\Delta})$ (a $G_{K,\Delta}$-stable lattice being given by $\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V \in \mathbf{Rep}^{\mathsf{reg}}_{\mathsf{B}_{\mathrm{dR},\Delta}}(G_{K,\Delta})$). Put

$$\mathsf{D}^+_{\mathsf{dif}}(V) = \big((\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V)^{H_{K,\Delta}}\big)_{\mathsf{f}}$$

and $\mathsf{D}_{\mathsf{dif}}(V) = \mathsf{l}_{\mathrm{dR},\Delta} \otimes_{\mathsf{l}^+_{\mathrm{dR},\Delta}} \mathsf{D}^+_{\mathsf{dif}}(V)$. By what precedes, this provides objects in $\mathscr{R}_{\mathsf{l}^+_{\mathrm{dR},\Delta}}$ and $\mathscr{R}^{\mathsf{reg}}_{\mathsf{l}_{\mathrm{dR},\Delta}}$ respectively.

We have $\mathsf{D}_{\mathsf{dif}}(V) = \mathsf{D}^+_{\mathsf{dif}}(V)\big[\frac{1}{t_\Delta}\big]$, $\mathsf{D}^+_{\mathsf{dif}}(V)^{\Gamma_{K,\Delta}} = (\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V)^{G_{K,\Delta}}$ and $\mathsf{D}_{\mathsf{dif}}(V)^{\Gamma_{K,\Delta}} = \mathsf{D}_{\mathrm{dR}}(V)$.

Proposition 5.18. *(cf [3, Proposition 7.1]) Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta})$. Then V is de Rham if and only if $\mathsf{D}_{\mathsf{dif}}(V)$ is trivial (as a module with connection).*

Proof. Assume V is de Rham: there exists $\underline{n} \in \mathbf{Z}^\Delta$ such that $V(\underline{n})$ has Hodge-Tate weight whose components are all non-positive, so that $\mathsf{D}_{\mathrm{dR}}(V(\underline{n})) = \mathsf{D}^+_{\mathrm{dR}}(V(\underline{n})) =: (\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V(\underline{n}))^{G_{K,\Delta}}$. The map $\alpha_{\mathrm{dR}}(V)$ is an isomorphism, hence

$$\alpha^+_{\mathrm{dR}} \colon \mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}^+_{\mathrm{dR}}(V(\underline{n})) \to \mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V(\underline{n})$$

is injective with cokernel killed by t_Δ. Taking invariants under $H_{K,\Delta}$, we get an injective map

$$g \colon \mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}^+_{\mathrm{dR}}(V(\underline{n})) \to (\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V(\underline{n}))^{H_{K,\Delta}}$$

whose cokernel is killed by some power of t_Δ. The commutative diagram

$$\begin{tikzcd}
\mathsf{I}^+_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}^+_{\mathrm{dR}}(V(\underline{n})) \arrow[r, "f"] \arrow[d, hook] & \mathsf{D}^+_{\mathsf{dif}}(V(\underline{n})) \arrow[d, hook] \\
\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}^+_{\mathrm{dR}}(V(\underline{n})) \arrow[r, "g"] & (\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V(\underline{n}))^{H_{K,\Delta}}
\end{tikzcd}$$

shows that f is injective. If $y \in \mathsf{D}^+_{\mathsf{dif}}(V(\underline{n}))$, there exists $N \in \mathbf{N}$ such that $t_\Delta^N y \in \mathsf{Im}(g) \cap \mathsf{D}^+_{\mathsf{dif}}(V(\underline{n}))$. By definition of the functor $X \mapsto X_{\mathsf{f}}$, this shows that $t_\Delta^N y \in \mathsf{I}^+_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}^+_{\mathsf{dif}}(V(\underline{n}))$. The cokernel of f is thus of t_Δ-torsion, so that f induces an isomorphism

$$\mathsf{I}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}^+_{\mathrm{dR}}(V(\underline{n})) \xrightarrow{\sim} \mathsf{D}_{\mathsf{dif}}(V(\underline{n})) = \mathsf{D}_{\mathsf{dif}}(V)$$

which implies that $\mathsf{D}_{\mathsf{dif}}(V)$ is trivial (as a module with connection).

Conversely, assume that $Y := \mathsf{D}_{\mathsf{dif}}(V)$ is trivial as a module with connection over $\mathsf{I}_{\mathrm{dR},\Delta}$: the natural map

$$\mathsf{I}_{\mathrm{dR},\Delta} \otimes_{K_{\Delta,\infty}} Y^{\nabla_Y=0} \to Y$$

is an isomorphism. As $Y^{\nabla_Y=0}$ is a $K_{\Delta,\infty}$-module of finite type, there exists $n \in \mathbf{N}$ such that $Y^{\nabla_Y=0} \subset t_\Delta^{-n}\, \mathsf{D}^+_{\mathsf{dif}}(V)$. Replacing V with $V(\underline{n})$ for an appropriate $\underline{n} \in \mathbf{Z}^\Delta$, we may asume that $n = 0$, so that $Y^{\nabla_Y=0} = \mathsf{D}^+_{\mathsf{dif}}(V)^{\nabla_Y=0} = K_{\Delta,\infty} \otimes_{K_\Delta} \mathsf{D}^+_{\mathrm{dR}}(V)$ (the last equality by Proposition 5.16). Extending the scalars from $K_{\Delta,\infty}$ to $\mathsf{B}_{\mathrm{dR},\Delta}$, we deduce that

$$\alpha_{\mathrm{dR}}(V)\colon\ \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{K_\Delta} \mathsf{D}^+_{\mathrm{dR}}(V) \to \mathsf{B}_{\mathrm{dR},\Delta} \otimes_{\mathbf{Q}_p} V$$

is an isomorphism, *i.e.* that V is de Rham. □

Now we relate our constructions to multivariate overconvergent (φ, Γ)-modules. These were constructed in [20] and [12], under the hypothesis that K is a finite extension of $\mathbf{Q}_p$, what we thus assume henceforth. We start by recalling the definitions and results we will need from [20] and [12]. In the classical, univariate case, put

$$\mathbf{A}_{F_0} = \Big\{ \sum_{n\in\mathbf{Z}} a_n \varpi^n\, ;\ (\forall n \in \mathbf{Z})\, a_n \in \mathsf{W}(k)\ \lim_{n\to-\infty} a_n = 0 \Big\}$$

where ϖ is seen as a variable. This is a Cohen ring for the field $\mathbf{E}_{F_0} = k(\!(\overline{\varpi})\!)$. We equip $\mathbf{A}_{F_0}$ with the commuting (semi-linear) Frobenius operator and

Γ_{F_0}-action given by

$$\varphi(\varpi) = (1+\varpi)^p - 1 \text{ and } \gamma(\varpi) = (1+\varpi)^{\chi(\gamma)} - 1.$$

Put $\mathbf{B}_{F_0} = \mathsf{Frac}(\mathbf{A}_{F_0}) = \mathbf{A}_{F_0}\left[\frac{1}{p}\right]$ and let $\mathbf{B}_{F_0}^{\mathsf{ur}}$ the maximal unramified extension of $\mathbf{B}_{F_0}$: this is a DVF with uniformizer p and whose residue field $\mathbf{E}$ is a separable closure of $\mathbf{E}_{F_0}$. We have an injective morphism $\mathbf{A}_{F_0} \to \widetilde{\mathbf{A}} := \mathsf{W}(C^\flat)$ sending ϖ to $[\varepsilon] - 1$. It extends into an injective map $\mathbf{A} \to \widetilde{\mathbf{A}}$, where $\mathbf{A}$ is the p-adic completion of the ring of integers of $\mathbf{B}_{F_0}^{\mathsf{ur}}$. There is a Frobenius operator φ and a action of G_{F_0} on $\mathbf{B} = \mathbf{A}\left[\frac{1}{p}\right]$ so that the previous map is G_{F_0}-equivariant and compatible with Frobenius. Moreover, there is an isomorphism $H_{F_0} \simeq \mathsf{Gal}(\mathbf{B} \,/\, \mathbf{B}_{F_0})$. As $H_K \leq H_{F_0}$, we put $\mathbf{B}_K = \mathbf{B}^{H_K}$: this is a finite Galois extension of $\mathbf{B}_0$, and there exists an element $\varpi_K \in \mathbf{B}_K$ such that its ring of integers

$$\mathbf{A}_K = \Big\{ \sum_{n\in\mathbf{Z}} a_n \varpi_K^n \,;\, (\forall n \in \mathbf{Z})\, a_n \in \mathsf{W}(k') \lim_{n\to-\infty} a_n = 0 \Big\}$$

where k' is the residue field of K_∞. This is a Cohen ring for the field $\mathbf{E}^{H_K} =: \mathbf{E}_K = k'(\!(\overline{\varpi}_K)\!)$.

By [15], the functor $T \mapsto (\mathbf{A} \otimes_{\mathbf{Z}_p} T)^{H_K}$ induces an equivalence between $\mathbf{Rep}_{\mathbf{Z}_p}(G_K)$ and the category $\mathbf{Mod}_{\mathbf{A}_K}^{\text{ét}}(\varphi, \Gamma)$ of étale (φ,Γ)-modules over $\mathbf{A}_K$, whose objects are $\mathbf{A}_K$-modules of finite type endowed with commuting and semi-linear Frobenius operator and Γ_K-action, such that linearization of the Frobenius operator is an isomorphism. By inverting p, there is a similar equivalence between the isogeny categories.

The multivariate generalization of Fontaine result was proved in [26]: let $\mathbf{A}_{K,\Delta}$ be the p-adic completion of the tensor product $\mathbf{A}_K \otimes_{\mathbf{Z}_p} \cdots \otimes_{\mathbf{Z}_p} \mathbf{A}_K$ where the copies of $\mathbf{A}_K$ are indexed by Δ (the copy of $\mathbf{A}_K$ of index $\alpha \in \Delta$ will be denoted $\mathbf{A}_{K,\alpha}$), and $\mathbf{B}_{K,\Delta} = \mathbf{A}_{K,\Delta}\left[\frac{1}{p}\right]$. We have $\mathbf{A}_{K,\Delta}/(p) =: \mathbf{E}_{K,\Delta} = \mathbf{E}_K \otimes_{\mathbf{F}_p} \cdots \otimes_{\mathbf{F}_p} \mathbf{E}_K$ (where the copies of $\mathbf{E}_K$ are indexed by Δ). For each $\alpha \in \Delta$, let φ_α and $\Gamma_{K,\alpha} = \iota_\alpha(\Gamma_K)$ denote the actions of φ and Γ_K on the factor of index α fixing the other factors. Denote by φ_Δ the monoid generated by the φ_α for $\alpha \in \Delta$. There exists a multivariate analogues of $\mathbf{A}$ and $\mathbf{B}$: let $\mathbf{A}_\Delta$ (resp. $\mathbf{B}_\Delta$) be the p-adic completion of $\varinjlim_F \mathbf{A}_{F,\Delta}$ where F runs over the finite subextensions of $\overline{K}/K$ (resp. $\mathbf{B}_\Delta = \mathbf{A}_\Delta\left[\frac{1}{p}\right]$). These rings are endowed with commuting actions of φ_Δ and $G_{K,\Delta}$, and $\mathbf{A}_\Delta/(p) \simeq \varinjlim_F \mathbf{E}_{F,\Delta}$. There is an equivalence of categories

$$\begin{aligned} \mathbf{Rep}_{\mathbf{Z}_p}(G_{K,\Delta}) &\to \mathbf{Mod}^{\text{ét}}_{\mathbf{A}_{K,\Delta}}(\varphi_\Delta, \Gamma_{K,\Delta}) \\ T &\mapsto (\mathbf{A}_\Delta \otimes_{\mathbf{Z}_p} T)^{H_{K,\Delta}} \end{aligned}$$

where $\mathbf{Rep}_{\mathbf{Z}_p}(G_{K,\Delta})$ is the category of $\mathbf{Z}_p$-modules of finite type endowed with a continuous linear action of $G_{K,\Delta}$, and $\mathbf{Mod}^{\text{ét}}_{\mathbf{A}_{K,\Delta}}(\varphi_\Delta, \Gamma_{K,\Delta})$ the category of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\mathbf{A}_{K,\Delta}$, whose objects are finitely generated projective $\mathbf{A}_{K,\Delta}$-modules with commuting semilinear actions of the φ_α for $\alpha \in \Delta$ and $\Gamma_{K,\Delta}$, and such that the linearization of φ_α is an isomorphism for all $\alpha \in \Delta$ (*cf* [12, §2.3] and [12, Theorem 4.1]). By inverting p, there is a similar equivalence $\mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta}) \to \mathbf{Mod}^{\text{ét}}_{\mathbf{B}_{K,\Delta}}(\varphi_\Delta, \Gamma_{K,\Delta})$ between the corresponding isogeny categories.

On the other hand, Fontaine result was refined by Cherbonnier-Colmez as follows. Let $v^\flat$ be the valuation on $C^\flat$ normalized by $v^\flat(\widetilde{p}) = 1$. If $r \in \mathbf{Q}_{>0}$, let $\widetilde{\mathbf{A}}^{(0,r]} \subset \widetilde{\mathbf{A}}$ be the subset made of those elements $z = \sum\limits_{m=0}^{\infty} p^m[z_m]$ (with $(z_m)_{m\in\mathbf{N}} \in (C^\flat)^{\mathbf{N}}$) such that $\lim\limits_{m\to\infty} rv^\flat(z_m) + m = +\infty$. Recall there are embeddings $\mathbf{A}_K \subset \mathbf{A} \hookrightarrow \widetilde{\mathbf{A}}$: put $\mathbf{A}_K^{(0,r]} = \mathbf{A}_K \cap \widetilde{\mathbf{A}}^{(0,r]}$ and $\mathbf{A}^{(0,r]} = \mathbf{A} \cap \widetilde{\mathbf{A}}^{(0,r]}$. Inverting p, we define analogues $\mathbf{B}_K^{(0,r]} \subset \mathbf{B}^{(0,r]}$. The subring of overconvergent elements in $\mathbf{A}_K$ (resp. $\mathbf{A}$) is $\mathbf{A}_K^\dagger := \bigcup\limits_{r>0} \mathbf{A}_K^{(0,r]}$ (resp. $\mathbf{A}^\dagger := \bigcup\limits_{r>0} \mathbf{A}^{(0,r]}$). Inverting p, we define analogues $\mathbf{B}_K^\dagger \subset \mathbf{B}^\dagger$ (these are fields). Those rings are stable under the action of G_K, and we have $(\mathbf{A}^{(0,r]})^{H_K} = \mathbf{A}_K^{(0,r]}$ and $(\mathbf{A}^\dagger)^{H_K} = \mathbf{A}_K^\dagger$. Moreover, we have $\varphi(\mathbf{A}^{(0,r]}) \subset \mathbf{A}^{(0,r/p]}$, so that $\mathbf{A}^\dagger$ and $\mathbf{A}_K^\dagger$ are stable under φ. Defining $\mathbf{Mod}^{\text{ét}}_{\mathbf{A}_K^\dagger}(\varphi, \Gamma)$ similarly as $\mathbf{Mod}^{\text{ét}}_{\mathbf{A}_K}(\varphi, \Gamma)$, we have:

Theorem 5.19. *([13, Proposition III.5.1 & corollaire III.5.2]) The functor*

$$\begin{aligned} \mathbf{Rep}_{\mathbf{Z}_p}(G_K) &\to \mathbf{Mod}^{\text{ét}}_{\mathbf{A}_K^\dagger}(\varphi, \Gamma) \\ T &\mapsto (\mathbf{A}^\dagger \otimes_{\mathbf{Z}_p} T)^{H_K} \end{aligned}$$

is an equivalence of categories.

This result was extended to the multivariate case in [20] and [12]. If $r \in \mathbf{R}_{>0}$, let $\mathbf{A}_{F_0,\Delta}^{(0,r]} \subset \mathbf{A}_{F_0,\Delta}$ be the subring made of those elements

$\sum_{\underline{n}\in\mathbf{Z}^\Delta}(a_{n_1}\varpi_{\alpha_1}^{n_1})\otimes\cdots\otimes(a_{n_\delta}\varpi_{\alpha_\delta}^{n_\delta})$ (with $(a_{n_1},\ldots,a_{n_\delta})\in\mathsf{W}(k)^\Delta$) such that

$$\lim_{|\underline{n}|\to\infty} v_p(a_{n_1}\cdots a_{n_\delta})+\tfrac{rp}{p-1}\min\{n_1,\ldots,n_\delta\}=+\infty.$$

Put $\mathbf{A}^\dagger_{F_0,\Delta}=\bigcup_{r\in\mathbf{R}_{>0}}\mathbf{A}^{(0,r]}_{F_0,\Delta}\subset\mathbf{A}_{F_0,\Delta}$ and $\mathbf{B}^\dagger_{F_0,\Delta}=\mathbf{A}^\dagger_{F_0,\Delta}\big[\frac{1}{p}\big]$. The subrings $\mathbf{A}^\dagger_{F_0,\Delta}$ and $\mathbf{B}^\dagger_{F_0,\Delta}$ of $\mathbf{B}_{F_0,\Delta}$ are stable by φ_Δ and $\Gamma_{F_0,\Delta}$. In this context, the analogue of $\mathbf{A}^\dagger$ is constructed as follows. For a finite subextension F of $\overline{K}/F_0$ and $r\in\mathbf{Q}_{>0}$, we put $\mathbf{A}^{(0,r]}_{F,\Delta,\circ}=\mathbf{A}^{(0,r]}_F\otimes_{\mathbf{Z}_p}\cdots\otimes_{\mathbf{Z}_p}\mathbf{A}^{(0,r]}_F$ (where the copies of $\mathbf{A}^{(0,r]}_F$ are indexed by Δ), and $\mathbf{A}^{(0,r]}_{F,\Delta}=\mathbf{A}^{(0,r]}_{F_0,\Delta}\otimes_{\mathbf{A}^{(0,r]}_{F_0,\Delta,\circ}}\mathbf{A}^{(0,r]}_{F,\Delta,\circ}$. We have $\mathbf{A}^{(0,r]}_{\Delta,\circ}=\varinjlim_F\mathbf{A}^{(0,r]}_{F,\Delta,\circ}=\mathbf{A}^{(0,r]}\otimes_{\mathbf{Z}_p}\cdots\otimes_{\mathbf{Z}_p}\mathbf{A}^{(0,r]}$. Put $\mathbf{A}^{(0,r]}_\Delta=\mathbf{A}^{(0,r]}_{F_0,\Delta}\otimes_{\mathbf{A}^{(0,r]}_{F_0,\Delta,\circ}}\mathbf{A}^{(0,r]}_{\Delta,\circ}$ and $\mathbf{A}^\dagger_\Delta=\bigcup_{r>0}\mathbf{A}^{(0,r]}_\Delta$. Inverting p we get rings $\mathbf{B}^{(0,r]}_\Delta\subset\mathbf{B}^\dagger_\Delta$. These rings admit an action of $G_{K,\Delta}$. By [20, Lemma 3.2.1], we have $(\mathbf{A}^\dagger_\Delta)^{H_{K,\Delta}}=\mathbf{A}^\dagger_{K,\Delta}=:\bigcup_{r>0}\mathbf{A}^{(0,r]}_{K,\Delta}$. Moreover, if $\alpha\in\Delta$, we have an operator $\varphi_\alpha\colon\mathbf{A}^{(0,r]}_\Delta\to\mathbf{A}^{(0,r/p]}_\Delta$, so that there is an action of φ_Δ on $\mathbf{A}^\dagger_\Delta$ and $\mathbf{B}^\dagger_\Delta$. The natural map $\mathbf{A}^\dagger_\Delta\to\mathbf{A}_\Delta$ (induced by the maps $\mathbf{A}^{(0,r]}_{\Delta,\circ}\to\mathbf{A}_\Delta$ extended by $\mathbf{A}^{(0,r]}_K$-linearity) is $G_{K,\Delta}$ and φ_Δ-equivariant. The generalization of Theorem 5.19 is:

Theorem 5.20. *([20, Corollary 3.4.4], [12, Theorem 6.15]) The functor*

$$\begin{aligned}\mathsf{D}^\dagger\colon\ \mathbf{Rep}_{\mathbf{Z}_p}(G_{K,\Delta})&\to\mathbf{Mod}^{\text{ét}}_{\mathbf{A}^\dagger_{K,\Delta}}(\varphi_\Delta,\Gamma_{K,\Delta})\\ T&\mapsto(\mathbf{A}^\dagger_\Delta\otimes_{\mathbf{Z}_p}T)^{H_K}\end{aligned}$$

where the target category is defined similarly as $\mathbf{Mod}^{\text{ét}}_{\mathbf{A}_{K,\Delta}}(\varphi_\Delta,\Gamma_{K,\Delta})$, is an equivalence of categories.

Moreover, by [20, Theorem 3.4.2] and [12, Theorem 6.14], base extension to $\mathbf{A}_{K,\Delta}$ over $\mathbf{A}^\dagger_{K,\Delta}$ induces an equivalence of categories

$$\mathbf{Mod}^{\text{ét}}_{\mathbf{A}^\dagger_{K,\Delta}}(\varphi_\Delta,\Gamma_{K,\Delta})\xrightarrow{\sim}\mathbf{Mod}^{\text{ét}}_{\mathbf{A}_{K,\Delta}}(\varphi_\Delta,\Gamma_{K,\Delta}).$$

Notation. If $T\in\mathbf{Rep}_{\mathbf{Z}_p}(G_{K,\Delta})$ and $r\in\mathbf{Q}_{>0}$, we put $\mathsf{D}^{(0,r]}(T)=\big(\mathbf{A}^{(0,r]}_\Delta\otimes_{\mathbf{Z}_p}T\big)^{H_{K,\Delta}}$. This is an $\mathbf{A}^{(0,r]}_{K,\Delta}$-module endowed with an action of $\Gamma_{K,\Delta}$. Moreover, for each $\alpha\in\Delta$, the operator φ_α induces a semi-linear map $\mathsf{D}^{(0,r]}(T)\to\mathsf{D}^{(0,r/p]}(T)$.

Lemma 5.21. *Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(G_{K,\Delta})$ be free of rank d. There exists $r_T \in \mathbf{Q}_{>0}$ such that for all $r \in \mathbf{Q} \cap]0, r_T]$, the natural map*

$$\alpha^{(0,r]}(T) \colon \mathbf{A}_{\Delta}^{(0,r]} \otimes_{\mathbf{A}_{K,\Delta}^{(0,r]}} \mathsf{D}^{(0,r]}(T) \to \mathbf{A}_{\Delta}^{(0,r]} \otimes_{\mathbf{Z}_p} T$$

is an isomorphism and $\mathsf{D}^{(0,r]}(T)$ is projective of rank d over $\mathbf{A}_{K,\Delta}^{(0,r]}$.

Proof. The $\mathbf{A}_{K,\Delta}^{\dagger}$-module $\mathsf{D}^{\dagger}(T)$ is projective of finite type: we can find a finite set $(a_i)_{1 \le i \le s}$ generating the unit ideal in $\mathbf{A}_{K,\Delta}^{\dagger}$ such that the localization $\mathsf{D}^{\dagger}(T)_{a_i}$ is free of rank $d = \mathsf{rk}_{\mathbf{Z}_p}(T)$ over $(\mathbf{A}_{K,\Delta}^{\dagger})_{a_i}$ for all $i \in \{1, \ldots, s\}$. We can choose $r_T \in \mathbf{Q}_{>0}$ small enough such that $a_i \in \mathbf{A}_{K,\Delta}^{(0,r_T]}$ and there exists a $(\mathbf{A}_{K,\Delta}^{\dagger})_{a_i}$-basis $(x_{i,j})_{1 \le j \le d}$ of $\mathsf{D}^{\dagger}(T)_{a_i}$ made of elements in $\mathsf{D}^{(0,r_T]}(T)_{a_i}$ for all $i \in \{1, \ldots, s\}$. Put $D_i = \bigoplus\limits_{j=1}^{d} \big(\mathbf{A}_{K,\Delta}^{(0,r]}(T)\big)_{a_i} x_{i,j} \subset \mathsf{D}^{(0,r]}(T)_{a_i}$. By Theorem 5.20, the natural map

$$\alpha^{\dagger}(T) \colon \mathbf{A}_{\Delta}^{\dagger} \otimes_{\mathbf{A}_{K,\Delta}^{\dagger}} \mathsf{D}^{\dagger}(T) \to \mathbf{A}_{\Delta}^{\dagger} \otimes_{\mathbf{Z}_p} T$$

is an isomorphism. Localizing, this provides an isomorphism $\big(\mathbf{A}_{\Delta}^{\dagger}\big)_{a_i} \otimes_{\mathbf{A}_{K,\Delta}^{\dagger}} \mathsf{D}^{\dagger}(T) \to \big(\mathbf{A}_{\Delta}^{\dagger}\big)_{a_i} \otimes_{\mathbf{Z}_p} T$, which implies that the localization $(\alpha^{(0,r_T]}(T))_{a_i} \colon \big(\mathbf{A}_{\Delta}^{(0,r]}\big)_{a_i} \otimes_{\mathbf{A}_{K,\Delta}^{(0,r]}} \mathsf{D}^{(0,r]}(T) \to \big(\mathbf{A}_{\Delta}^{(0,r]}\big)_{a_i} \otimes_{\mathbf{Z}_p} T$ is injective. In the basis $(1 \otimes x_{i,j})_{1 \le j \le d}$ and the basis induced by any basis of T over $\mathbf{Z}_p$, this isomorphism is given by a matrix $M_i \in \mathsf{GL}_d\big(\big(\mathbf{A}_{\Delta}^{\dagger}\big)_{a_i}\big)$. Shrinking r_T further if necessary, we may assume that $M_i \in \mathsf{GL}_d\big(\big(\mathbf{A}_{\Delta}^{(0,r_T]}\big)_{a_i}\big)$. This means that the composite map

$$\begin{aligned}\big(\mathbf{A}_{\Delta}^{(0,r_T]}\big)_{a_i} \otimes_{(\mathbf{A}_{K,\Delta}^{(0,r_T]})_{a_i}} D_i \to \big(\mathbf{A}_{\Delta}^{(0,r_T]}\big)_{a_i} &\otimes_{\mathbf{A}_{K,\Delta}^{(0,r_T]}} \mathsf{D}^{(0,r_T]}(T) \\ &\xrightarrow{(\alpha^{(0,r_T]}(T))_{a_i}} \big(\mathbf{A}_{\Delta}^{(0,r_T]}\big)_{a_i} \otimes_{\mathbf{Z}_p} T\end{aligned}$$

is an isomorphism. This implies that $(\alpha^{(0,r_T]}(T))_{a_i}$ is surjective, hence an isomorphism, so that the map $\big(\mathbf{A}_{\Delta}^{(0,r_T]}\big)_{a_i} \otimes_{\mathbf{A}_{K,\Delta}^{(0,r_T]}} D_i \to \big(\mathbf{A}_{\Delta}^{(0,r_T]}\big)_{a_i} \otimes_{\mathbf{A}_{K,\Delta}^{(0,r_T]}} \mathsf{D}^{(0,r_T]}(T)$ is an isomorphism as well. Taking invariants under $H_{K,\Delta}$ implies that the map $D_i \to \big(\mathbf{A}_{K,\Delta}^{(0,r_T]}\big)_{a_i} \otimes_{\mathbf{A}_{K,\Delta}^{(0,r_T]}} \mathsf{D}^{(0,r_T]}(T)$ is an isomorphism. As this holds for all $i \in \{1, \ldots, s\}$, this shows that $\alpha^{(0,r_T]}(T)$ is an isomorphism and that $\mathsf{D}^{(0,r_T]}(T)$ is projective of rank d over $\mathbf{A}_{K,\Delta}^{(0,r_T]}$. This implies that the similar statement holds when r_T is replaced by any $r \in \mathbf{Q} \cap]0, r_T]$. $\square$

If $n \in \mathbf{N}_{>0}$ and $r \geq r_n := \frac{1}{(p-1)p^{n-1}}$, there is an injective map $i_n \colon \widetilde{\mathbf{A}}^{(0,r]} \to \mathsf{B}^+_{\mathrm{dR}}$ such that $i_n(\varpi) = [\varepsilon^{1/p^n}] - 1 = \varepsilon^{(n)} \exp(t/p^n) - 1$ and $i_n(\mathbf{A}^{(0,r]}_K) \subset K_n[\![t]\!]$ (*cf* [14, Proposition III.2.1]). The tensor product of these maps, indexed by Δ, induces a ring homomorphism

$$i_{n,\Delta} \colon \mathbf{A}^{(0,r]}_{\Delta,\circ} \to \bigotimes_{\alpha \in \Delta} \mathsf{B}^+_{\mathrm{dR}} \to \mathsf{B}^+_{\mathrm{dR},\Delta}$$

(note that it is not injective in general, since the tensor product in the LHS is taken over $\mathbf{Z}_p$ whereas it is taken over F_0 in the RHS). It restricts into a ring homomorphism

$$i_{n,\Delta} \colon \mathbf{A}^{(0,r]}_{K,\Delta,\circ} \to \bigotimes_{\alpha \in \Delta} K_n[\![t_\alpha]\!] \to \mathsf{I}^+_{\mathrm{dR},\Delta}.$$

Lemma 5.22. *If $r > \max\{r_n, \delta p^{-n}\}$, the restriction of $i_{n,\Delta}$ to $\mathbf{A}^{(0,r]}_{F_0,\Delta,\circ}$ extends into a map $\mathbf{A}^{(0,r]}_{F_0,\Delta} \to F_{0,\infty}[\![t_\alpha]\!] \to \mathsf{I}^+_{\mathrm{dR},\Delta}$.*

Proof. This is checked modulo $\mathsf{Fil}^{s+1}\,\mathsf{I}^+_{\mathrm{dR},\Delta}$ for all $s \in \mathbf{N}$. If $x = \sum_{\underline{n} \in \mathbf{Z}^\Delta} (a_{n_1}\varpi^{n_1}) \otimes \cdots \otimes (a_{n_\delta}\varpi^{n_\delta}) \in \mathbf{A}^{(0,r]}_{F_0,\Delta}$, we have to check that the series $i_{n,\Delta}(x) = \sum_{\underline{n} \in \mathbf{Z}^\Delta} a_{n_1} \cdots a_{n_\delta} \prod_{i=1}^{\delta} \left(\varepsilon^{(n)} \exp(t_{\alpha_i}/p^n) - 1\right)^{n_i}$ converges in $F_{0,\infty}[t_\alpha]_{\alpha\in\Delta}/(t_\alpha)^{s+1}_{\alpha\in\Delta}$ for the p-adic topology. For each $\underline{n} \in \mathbf{Z}^\Delta$, we have

$$\prod_{i=1}^{\delta} \left(\varepsilon^{(n)} \exp(t_{\alpha_i}/p^n) - 1\right)^{n_i} = \prod_{i=1}^{\delta} \Big(\sum_{k_i=0}^{n_i} (-1)^{n_i-k_i} \left(\varepsilon^{(n)}\right)^{k_i} \tbinom{n_i}{k_i} \exp(k_i t_{\alpha_i}/p^n) \Big)$$
$$= \prod_{i=1}^{\delta} \Big((\varepsilon^{(n)} - 1)^{n_i} + \sum_{k_i=0}^{n_i} (-1)^{n_i-k_i} \left(\varepsilon^{(n)}\right)^{k_i} \tbinom{n_i}{k_i} (\exp(k_i t_{\alpha_i}/p^n) - 1) \Big)$$

If $\underline{m} = (m_1, \ldots, m_\delta) \in \mathbf{N}^\Delta$, the coefficient of $t_{\alpha_1}^{m_1} \cdots t_{\alpha_\delta}^{m_\delta}$ in the latter expression is

$$c_{n,\underline{m}} := (\varepsilon^{(n)} - 1)^{\sum\limits_{\substack{1\leq i\leq\delta \\ m_i = 0}} n_i} \prod_{\substack{1\leq i\leq\delta \\ m_i \neq 0}} \Big(\sum_{k_i=0}^{n_i} (-1)^{n_i-k_i} \left(\varepsilon^{(n)}\right)^{k_i} \tbinom{n_i}{k_i} \left(\tfrac{k_i}{p^n}\right)^{m_i} \Big)$$

whose valuation is larger that $\frac{1}{(p-1)p^{n-1}}\Big(\sum\limits_{\substack{1\le i\le\delta\\ m_i=0}} n_i\Big) - n|\underline{m}|$. The coefficient of $t_{\alpha_1}^{m_1}\cdots t_{\alpha_\delta}^{m_\delta}$ in $i_{n,\Delta}(x)$ is the sum of the series $\sum\limits_{\underline{n}\in\mathbf{Z}^\Delta} a_{n_1}\cdots a_{n_\delta}c_{n,\underline{m}}$. If $\underline{n}\in\mathbf{Z}^\Delta$, we have $v_p(a_{n_1}\cdots a_{n_\delta}c_{n,\underline{m}}) \ge \frac{1}{(p-1)p^{n-1}}\Big(\sum\limits_{\substack{1\le i\le\delta\\ m_i=0}} n_i\Big) - n|\underline{m}|$ if $\mu_{\underline{n}} := \min\{n_1,\ldots,n_\delta) \ge 0$, and this goes to $+\infty$ when $|\underline{n}|\to\infty$. Assume $\mu_{\underline{n}}<0$: we have

$$\begin{aligned}
v_p(a_{n_1}\cdots a_{n_\delta}c_{n,\underline{m}}) &= v_p(a_{n_1}\cdots a_{n_\delta}) + \tfrac{rp}{p-1}\mu_{\underline{n}} + v_p(c_{n,\underline{m}}) - \tfrac{rp}{p-1}\mu_{\underline{n}}\\
&\ge v_p(a_{n_1}\cdots a_{n_\delta}) + \tfrac{rp}{p-1}\mu_{\underline{n}} + \tfrac{1}{(p-1)p^{n-1}}\sum_{\substack{1\le i\le\delta\\ m_i=0}} n_i - n|\underline{m}| - \tfrac{rp}{p-1}\mu_{\underline{n}}\\
&\ge v_p(a_{n_1}\cdots a_{n_\delta}) + \tfrac{rp}{p-1}\mu_{\underline{n}} + \tfrac{1}{(p-1)p^{n-1}}\delta\mu_{\underline{n}} - n|\underline{m}| - \tfrac{rp}{p-1}\mu_{\underline{n}}\\
&\ge v_p(a_{n_1}\cdots a_{n_\delta}) + \tfrac{rp}{p-1}\mu_{\underline{n}} - n|\underline{m}| + \tfrac{p}{p-1}\big(\tfrac{\delta}{p^n} - r\big)\mu_{\underline{n}}\\
&\ge v_p(a_{n_1}\cdots a_{n_\delta}) + \tfrac{rp}{p-1}\mu_{\underline{n}} - n|\underline{m}|.
\end{aligned}$$

As $\lim\limits_{|\underline{n}|\to\infty} v_p(a_{n_1}\cdots a_{n_\delta}) + \frac{rp}{p-1}\min\{n_1,\ldots,n_\delta\} = +\infty$ by hypothesis, this shows that the series indeed converges. $\square$

If $r>\max\{r_n,\delta p^{-n}\}$, Lemma 5.22 implies that $\mathsf{B}^+_{\mathrm{dR},\Delta}$ is equipped with a $\mathbf{A}^{(0,r]}_{F_0,\Delta}$-algebra structure: the map $i_{n,\Delta}\colon \mathbf{A}^{(0,r]}_{\Delta,\circ}\to\mathsf{B}^+_{\mathrm{dR},\Delta}$ extends into a ring homomorphism

$$i_{n,\Delta}\colon \mathbf{A}^{(0,r]}_{\Delta}\to\mathsf{B}^+_{\mathrm{dR},\Delta}\,.$$

Theorem 5.23. *Let $T\in\mathbf{Rep}_{\mathbf{Z}_p}(G_{K,\Delta})$. If $r\in\mathbf{Q}_{>0}$ is small enough, there is a $\Gamma_{K,\Delta}$-equivariant isomorphism of $\mathsf{I}^+_{\mathrm{dR},\Delta}$-modules*

$$\mathsf{I}_{\mathrm{dR},\Delta}\otimes_{\mathbf{A}^{(0,r]}_{K,\Delta}}\mathsf{D}^{(0,r]}(T)\xrightarrow{\sim}\mathsf{D}_{\mathsf{dif}}\big(T\big[\tfrac{1}{p}\big]\big).$$

Proof. By Lemma 5.21, there exists $r_T\in\mathbf{Q}_{>0}$ such that for all $r\in\mathbf{Q}\cap]0,r_T]$, the natural map

$$\mathbf{A}^{(0,r]}_{\Delta}\otimes_{\mathbf{A}^{(0,r]}_{K,\Delta}}\mathsf{D}^{(0,r]}(T)\to\mathbf{A}^{(0,r]}_{\Delta}\otimes_{\mathbf{Z}_p}T$$

is a $G_{K,\Delta}$-equivariant isomorphism. Take $n\in\mathbf{N}_{>0}$ large enough such that $\max\{r_n,\delta p^{-n}\}\le r_T$, and assume that $r\in\mathbf{Q}_{>0}$ is such that $\max\{r_n,\delta p^{-n}\}\le$

$r \leq r_T$. Extending the scalars to $\mathsf{B}^+_{\mathrm{dR},\Delta}$ via $i_{n,\Delta}$ provides a $G_{K,\Delta}$-equivariant isomorphism

$$\mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{A}^{(0,r]}_{K,\Delta}} \mathsf{D}^{(0,r]}(T) \to \mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{Z}_p} T \simeq \mathsf{B}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} \mathsf{D}^+_{\mathsf{dif}}(V)$$

where $V = T\left[\frac{1}{p}\right] \in \mathbf{Rep}_{\mathbf{Q}_p}(G_{K,\Delta})$. Taking invariants under $H_{K,\Delta}$ gives a $\Gamma_{K,\Delta}$-equivariant isomorphism

$$\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{A}^{(0,r]}_{K,\Delta}} \mathsf{D}^{(0,r]}(T) \xrightarrow{\sim} \mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathsf{I}^+_{\mathrm{dR},\Delta}} \mathsf{D}^+_{\mathsf{dif}}(V).$$

Applying the functor $X \mapsto X_{\mathsf{f}}$ provides an isomorphism

$$\left(\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{A}^{(0,r]}_{K,\Delta}} \mathsf{D}^{(0,r]}(T)\right)_{\mathsf{f}} \xrightarrow{\sim} \mathsf{D}^+_{\mathsf{dif}}(V)$$

and it remains to show that $\left(\mathsf{L}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{A}^{(0,r]}_{K,\Delta}} \mathsf{D}^{(0,r]}(T)\right)_{\mathsf{f}} = \mathsf{I}^+_{\mathrm{dR},\Delta} \otimes_{\mathbf{A}^{(0,r]}_{K,\Delta}} \mathsf{D}^{(0,r]}(T)$. As $\mathsf{D}^{(0,r]}(T)$ is projective of finite rank over $\mathbf{A}^{(0,r]}_{K,\Delta}$, it is enough to show that $\mathsf{D}^{(0,r]}(T)$ is mapped to $\mathsf{D}^+_{\mathsf{dif}}(V)$ by $i_{n,\Delta}$. By [20, Lemma 3.2.4], we have $\mathsf{D}^\dagger(T) = \mathbf{A}^\dagger_{F_0,\Delta} \otimes_{\mathbf{A}^\dagger_{F_0,\Delta,\circ}} \left(\mathbf{A}^\dagger_{\Delta,\circ} \otimes_{\mathbf{Z}_p} T\right)^{H_{K,\Delta}}$: similarly, we have $\mathsf{D}^{(0,r]}(T) = \mathbf{A}^{(0,r]}_{F_0,\Delta} \otimes_{\mathbf{A}^{(0,r]}_{F_0,\Delta,\circ}} \left(\mathbf{A}^{(0,r]}_{\Delta,\circ} \otimes_{\mathbf{Z}_p} T\right)^{H_{K,\Delta}}$, so we are reduced to check that $\left(\mathbf{A}^{(0,r]}_{\Delta,\circ} \otimes_{\mathbf{Z}_p} T\right)^{H_{K,\Delta}}$ maps to $\mathsf{D}^+_{\mathsf{dif}}(V)$ by $i_{n,\Delta}$. Working componentwise, this follows from [5, Proposition 5.7]. $\square$

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