Journal of Number Theory 264 (2024) 27–40

Contents lists available at ScienceDirect

Journal of Number Theory

journal homepage: www.elsevier.com/locate/jnt

General Section

A conjecture of Flach and Morin

Bruno Chiarellotto [1], Nicola Mazzari [*,3], Yukihide Nakada [1,2]

Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova, Via Trieste 63, 35121, Padova, Italy

ARTICLE INFO

Article history:
Received 14 June 2023
Received in revised form 2 May 2024
Accepted 3 May 2024
Available online 26 June 2024
Communicated by A. Pal

In memory of Pierre Berthelot

MSC:
primary 14F30
secondary 14G22, 11G25

Keywords:
Log-crystalline cohomology
Monodromy
Rigid cohomology
p-adic cohomology

ABSTRACT

A conjecture recently stated by Flach and Morin relates the action of the monodromy on the Galois invariant part of the p-adic Beilinson–Hyodo–Kato cohomology of the generic fiber of a scheme defined over a DVR of mixed characteristic to (the cohomology of) its special fiber. We prove the conjecture in the case that the special fiber of the given arithmetic scheme is also a fiber of a geometric family over a curve in positive characteristic.

1. Introduction

Fix a prime number p. Let $K/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_K$ and residue field k. Let $W = W(k)$ be the ring of Witt vectors of k and let K_0 be its fraction

* Corresponding author.
E-mail addresses: chiarbru@math.unipd.it (B. Chiarellotto), nicola.mazzari@unipd.it (N. Mazzari), yukihide.nakada@unipd.it (Y. Nakada).

[1] Supported by grant MIUR-PRIN2017 "Geometric, Algebraic, and Analytic Methods in Arithmetic".
[2] Supported by INdAM grant INdAM-DP-COFUND-2015.
[3] Supported by BIRD 2022 "Arithmetic cohomology theories".

https://doi.org/10.1016/j.jnt.2024.05.013

field. We write $S = \mathrm{Spec}(W)$, and in the context of log structures we write $S^\times$ (resp. $S^\varnothing$) for S equipped with the canonical log structure $1 \mapsto 0$ (resp. the trivial log structure).

Let $f : X \to \mathrm{Spec}(\mathcal{O}_K)$ be a flat, projective morphism of relative dimension n. We write X_s and $X_{\overline{s}}$ for its special and geometric special fiber, respectively, and X_η and $X_{\overline{\eta}}$ for its generic and geometric generic fibers, respectively. In [11, §7], Flach and Morin speculate on the relationship between the geometric cohomology theories of the generic and special fibers. The geometric cohomology groups of the generic fiber

$$X \mapsto H^{B,i}_{HK}(X_{\overline{\eta},h})$$

are the *Beilinson–Hyodo–Kato* ones, considered in [17] and taking values in the category of (ϕ, N, G_K)-modules (for more on this structure see, for example, [4]). This cohomology theory was defined by Beilinson for any K-scheme Z, neither smooth nor proper, using h-descent [1].[4]

Moreover, when the morphism f is smooth or log-smooth, there are canonical isomorphisms between Beilinson–Hyodo–Kato cohomology and the cohomology of the geometric special fiber: namely, if f is smooth then it coincides with crystalline cohomology of the geometric special fiber $X_{\overline{s}}$ and if f is log-smooth then it coincides with the log-crystalline (Hyodo–Kato) cohomology of $X_{\overline{s}}$ (see [11, §7.2]).

The p-adic Weil cohomology theory for varieties Y/k is rigid cohomology (with coefficients in K_0)

$$Y \mapsto H^i_{\mathrm{rig}}(Y)$$

taking values in the category of φ-modules, i.e., finite-dimensional K_0-vector spaces with a Frobenius-semilinear endomorphism φ. In their article, Flach and Morin conjecture the following relationship between the two cohomology theories:

Conjecture. *([11, Conjecture 7.15]) For regular X of absolute dimension $d = n+1$ there is an exact triangle in the category of φ-modules*

$$R\Gamma_{\mathrm{rig}}(X_s) \xrightarrow{\mathrm{sp}} \left[R\Gamma^B_{HK}(X_{\overline{\eta},h})^{G_K} \xrightarrow{N} R\Gamma^B_{HK}(X_{\overline{\eta},h})(-1)^{G_K}\right] \xrightarrow{\mathrm{sp}'} R\Gamma^*_{\mathrm{rig}}(X_s)(-d)[-2d+1] \to,$$

where sp *induces the specialization map defined in [22] and* sp′ *is the composite of the Poincaré duality isomorphism*

[4] The Beilinson–Hyodo–Kato is related to p-adic étale cohomology via the Fontaine functor D_{pst}, namely

$$H^{B,i}_{HK}(X_{\overline{\eta},h}) \cong D_{pst}(H^i(X_{\overline{\eta}}, \mathbb{Q}_p)) := \mathrm{colim}_{H \le G_K,\mathrm{open}}(\mathbb{B}_{st} \otimes_{K_0} H^i(X_{\overline{\eta}}, \mathbb{Q}_p))^H ,$$

but we will not use this fact.

$$R\Gamma^B_{HK}(X_{\overline{\eta},h})(-1) \cong R\Gamma^B_{HK}(X_{\overline{\eta},h})^*(-d)[-2d+2]$$

on $X_{\overline{\eta}}$ *and* sp^**.*

We can reformulate this conjecture in the case where f is log-smooth using existing comparisons between the cohomology of special and generic fibers. Namely, under this extra condition, we already have a comparison between the cohomology of the special and generic fibers: Tsuji's theorem [20, Theorem 0.2] provides a canonical isomorphism

$$H^{B,i}_{HK}(X_{\overline{\eta},h})^{G_K} \cong H^i_{\text{log-crys}}(X_k/S^\times).$$

We thereby obtain the following crystalline realization of this exact triangle in the category of φ-modules:

$$\begin{aligned} R\Gamma_{\mathrm{rig}}(X_s) \to \Big[R\Gamma_{\text{log-crys}}(X_s/S^\times) \xrightarrow{N} R\Gamma_{\text{log-crys}}(X_s/S^\times)(-1)\Big] \to& \\ R\Gamma^*_{\mathrm{rig}}(X_s)(-n-1)[-2n-1] \to .& \end{aligned} \tag{1}$$

Thus, in the log-smooth case, the conjecture describes the monodromy operator on the log-crystalline cohomology of X_s in terms of rigid cohomology and its Poincaré dual.

Since the triangle (1) no longer involves the generic fiber X_η or the valuation ring $\mathcal{O}_K$, one can untwine the triangle from its original context and ask under what conditions on a k-scheme X_s such an exact triangle exists. The main result of this paper is the following theorem, which states that such an exact triangle exists when X_s is the special fiber, not of an arithmetic family $f : X \to \mathrm{Spec}(\mathcal{O}_K)$, but a geometric family $f : X \to C$ where C is a curve over k:

Theorem 1. *Let $f : X \to C$ be a proper, flat, generically smooth morphism over k of relative dimension n, where C is a smooth curve and X is smooth. Assume that for some k-rational point $s \in C$ the fiber X_s is a normal crossing divisor in X. Endow X with the log structure given by the divisor X_s and endow X_s itself with the pullback log-structure. Then there is an exact triangle*

$$\begin{tikzcd} & R\Gamma^*_{\mathrm{rig}}(X_s)(-n-1)[-2n-1] \arrow[dl] & \\ R\Gamma_{\mathrm{rig}}(X_s) \arrow[rr] & & \Big[R\Gamma_{\text{log-crys}}(X_s/S^\times) \xrightarrow{N} R\Gamma_{\text{log-crys}}(X_s/S^\times)(-1)\Big] \arrow[ul] \end{tikzcd}$$

in the derived category of φ-modules.

Our driving methodology is to adapt Chiarellotto and Tsuzuki's proof [9] of the existence and exactness of the Clemens–Schmid sequence

$$\cdots \to H^m_{\mathrm{rig}}(X_s) \xrightarrow{\gamma} H^m_{\text{log-crys}}((X_s, M_s)/S^\times) \otimes K \xrightarrow{N_m}$$
$$H^m_{\text{log-crys}}((X_s, M_s)/S^\times) \otimes K(-1) \xrightarrow{\delta} H^{m+2}_{X_s,\mathrm{rig}}(X) \xrightarrow{\alpha} H^{m+2}_{\mathrm{rig}}(X_s) \to \cdots$$

together with the theory of log-convergent and log-rigid cohomology. In [9] the authors need to restrict themselves to the case of finite fields to prove their main result because they use [7]. Since in the present article we do not need to compare the monodromy and the weight filtrations, we can avoid this restriction. Another difference from [9] is that we prove a result at the level of the derived category of complexes and not just in cohomology.

The similarity between the Clemens–Schmid sequence and the Flach-Morin triangle is clear in light of Berthelot's Poincaré duality [2], which states that the dual of rigid cohomology is given by the rigid cohomology with compact support

$$R\Gamma_{X_s,\mathrm{rig}}(Y) \cong R\Gamma^*_{\mathrm{rig}}(X_s)(-n-1)[-2n-2]$$

where Y is a smooth k-scheme admitting a closed immersion $X_s \hookrightarrow Y$; in our context we may choose, in particular, $Y = X$.

We follow their general idea of linking the localization triangle for rigid cohomology with respect to the closed subscheme X_s with the canonical exact triangle for the monodromy operator in log-crystalline cohomology, using the fact that the rigid cohomology of the open complement of a closed subscheme can be computed using logarithmic structures.

We work with log-rigid cohomology (see § 2) in place of log-crystalline cohomology because of the flexibility of the former. This is possible because log-crystalline and log-rigid cohomology, and their respective monodromy operators, agree in the proper and log-smooth case (Lemma 4).

After the submission of this article, Binda–Gallauer–Vezzani showed in [5] how to deduce the Clemens–Schmid sequence and the Flach–Morin conjecture with motivic methods. While their approach proves the general case, our method is more explicit.

Acknowledgments. The authors thank Veronika Ertl for insightful conversations. They also warmly thank the referee for the valuable remarks that significantly improved the quality of the article.

2. Review of rigid cohomology

We begin by reviewing the basic concepts of rigid cohomology that we will need in the sequel. For the moment, we forget about log structures. A *frame* $(X \subseteq Y \subseteq \mathfrak{P})$ (see [15, Definition 3.1.5]) is a sequence of inclusions

$$X \hookrightarrow Y \hookrightarrow \mathfrak{P}$$

where $X \hookrightarrow Y$ is an open immersion of k-varieties and $Y \hookrightarrow \mathfrak{P}$ is a closed immersion in a formal W-scheme $\mathfrak{P}$.

Fix a frame $(X \subseteq X' \subseteq \mathfrak{P})$ where X, X', and $\mathfrak{P}$ are separated and locally of finite type and where $\mathfrak{P}$ is smooth in a neighborhood of X. The *rigid cohomology of the pair* (X, X') is defined as

$$R\Gamma_{\mathrm{rig}}((X, X')) := R\Gamma(]X'[_{\mathfrak{P}}, j_X^\dagger \Omega^\bullet_{]X'[_{\mathfrak{P}}}) \ , \tag{2}$$

where $j_X^\dagger$ is the functor of overconvergent sections [15, p.129] and is denoted by $j^\dagger_{]X[_{\mathfrak{P}}}$ in [9]. In general, for a rigid analytic variety V and an admissible open $T \subset V$ it is possible to define the functor $j^\dagger_{V\setminus T}$ of overconvergent sections along T. When $V \subset]X'[_{\mathfrak{P}}$ is a strict neighborhood of $]X[_{\mathfrak{P}}$ and $T = V \cap (]X'[_{\mathfrak{P}} \setminus]X[_{\mathfrak{P}})$ we have $j_X^\dagger = j^\dagger_{V\setminus T}$. *A priori* the definition in (2) also depends on the formal scheme $\mathfrak{P}$, but it can be shown (see [15, 7.4.2], for example) that it depends (up to quasi-isomorphism) only on the immersion $X \hookrightarrow X'$.

The two important cases of this construction that we use are the following. First, the *convergent cohomology* of X is defined to be

$$R\Gamma_{\mathrm{conv}}(X) := R\Gamma_{\mathrm{rig}}((X, X)).$$

Second, if $X \hookrightarrow \overline{X}$ is an open immersion with $\overline{X}$ proper, the *rigid cohomology* of X is defined to be

$$R\Gamma_{\mathrm{rig}}(X) := R\Gamma_{\mathrm{rig}}((X, \overline{X})).$$

It is a fundamental result of the rigid cohomology theory established by Berthelot that this definition is not only independent of the formal scheme $\mathfrak{P}$, but also of the compactification $\overline{X}$ (see [15, Proposition 8.2.1]). Note that if X is already proper, then the convergent cohomology coincides with the rigid cohomology.

To understand the dual $R\Gamma^*_{\mathrm{rig}}(X_s)$, we will also need the notion of rigid cohomology with support in a closed subset.

Definition 2. Let X be a k-scheme, $Z \subseteq X$ a closed subscheme, and fix a frame $(X \subseteq \overline{X} \subseteq \mathfrak{P})$ where $\overline{X}$ is proper and $\mathfrak{P}$ is smooth in a neighborhood of X. We define

$$R\Gamma_{Z,\mathrm{rig}}(X) := R\Gamma(]\overline{X}[_P, \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{P}}} j_X^\dagger \Omega^\bullet_{]\overline{X}[_{\mathfrak{P}}})$$

to be the *rigid cohomology of X with support in Z*. (see [2] or [15, Definition 6.3.1])

Its relation to standard rigid cohomology is as follows. As an immediate consequence of the definition (see [15, Proposition 5.2.4 (ii)]) we have, for any sheaf $\mathcal{E}$ on $]\overline{X}[_{\mathfrak{P}}$, the following exact sequence

$$0 \to \underline{\Gamma}^\dagger_Z j^\dagger_X \mathcal{E} \to j^\dagger_X \mathcal{E} \to j^\dagger_{]\overline{X}[_{\mathfrak{P}} \setminus]Z[_{\mathfrak{P}}} j^\dagger_X \mathcal{E} \to 0 \; .$$

But by [15, Proposition 5.1.7], we get the exact sequence

$$0 \to \underline{\Gamma}^\dagger_Z j^\dagger_X \mathcal{E} \to j^\dagger_X \mathcal{E} \to j^\dagger_U \mathcal{E} \to 0 \; ,$$

where $U = X \setminus Z$. In this way we obtain an exact sequence

$$0 \to \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{P}}} j^\dagger_X \Omega^\bullet_{]\overline{X}[_{\mathfrak{P}}} \to j^\dagger_X \Omega^\bullet_{]\overline{X}[_{\mathfrak{P}}} \to j^\dagger_U \Omega^\bullet_{]\overline{X}[_{\mathfrak{P}}} \to 0$$

which induces in cohomology the *localization triangle*

$$R\Gamma_{Z,\mathrm{rig}}(X) \to R\Gamma_{\mathrm{rig}}(X) \to R\Gamma_{\mathrm{rig}}(U) \xrightarrow{+} .$$

There is a theory of log-rigid cohomology, described by Große-Klönne (see, for example, [12]) that generalizes the log-crystalline cohomology of Hyodo and Kato in the same way that rigid cohomology generalizes crystalline cohomology. Here, we give a brief overview and refer the reader to [12, §1.3] for precise definitions.

Write $\mathfrak{S} := \mathrm{Spf}(W)$ and let $\mathfrak{S}^\times$ (resp. $\mathfrak{S}^\varnothing$) denote the weak formal log-scheme $(\mathfrak{S}, 1 \mapsto 0)$ (resp. $(\mathfrak{S}, \text{trivial})$). Let X be a fine log-scheme over the log point $s^\times = (\mathrm{Spec}(k), 1 \mapsto 0)$. One can choose an open covering $X = \bigcup_{i \in I} V_i$ with exact closed immersions $V_i \hookrightarrow \mathfrak{V}_i$ and for each $H \subseteq I$ let $\mathfrak{V}_H$ be an exactification

$$V_H := \bigcap_{i \in H} V_i \to \mathfrak{V}_H \to \varprojlim_{i \in H} \mathfrak{V}_i \; ,$$

where the inverse limit is taken in the category of weak formal schemes over $\mathfrak{S}$. From these weak formal embeddings we can canonically construct a simplicial dagger space $]V_\bullet[_{\mathfrak{V}_\bullet} := (]V_H[_{\mathfrak{V}_H})_{H \subseteq I}$. The cohomology of the corresponding de Rham complex

$$R\Gamma_{\text{log-rig}}(X/\mathfrak{S}^\times) := R\Gamma(]V_\bullet[_{\mathfrak{V}_\bullet}, \Omega^\bullet_{]V_\bullet[_{\mathfrak{V}_\bullet}})$$

is defined as the *log-rigid cohomology of* X with respect to $\mathfrak{S}^\times$. It can be shown to be independent of the covering $X = \bigcup_i V_i$.[5]

One can also define a logarithmic equivalent of convergent cohomology for a fine log scheme X/k, called log-convergent cohomology, which we denote by $R\Gamma_{\text{log-conv}}(X/\mathfrak{S}^\times)$. It is generally defined formally as the cohomology of the trivial isocrystal on the log-convergent site ([19, §2.1]), but it also admits an interpretation through the cohomology of a suitable logarithmic de Rham complex [19, §2.1, Corollary 2.3.9]. In fact, by replacing weak formal schemes and dagger spaces with formal schemes and rigid spaces in the

[5] Although we will not use it, it might be of interest for the reader that the complex $R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^\times)$ together with its monodromy operator can be calculated using the overconvergent logarithmic de Rham–Witt complex of Gregory–Langer [13].

definition of log-rigid cohomology, one recovers Shiho's log-convergent cohomology [12, §1.5]. In particular, if X is proper, then log-convergent cohomology coincides with log-rigid cohomology. If X is additionally log-smooth, these two cohomologies also coincide with log-crystalline cohomology [12, pp. 401].

Finally, we can also repeat the construction of log-rigid cohomology with $\mathfrak{S}^{\varnothing}$ to obtain cohomology groups $R\Gamma_{\text{log-rig}}(X/\mathfrak{S}^{\varnothing})$. If now X is a k-scheme equipped with the trivial log structure, not necessarily proper, we have an isomorphism (see [12, pp. 401])

$$R\Gamma_{\text{log-rig}}(X/\mathfrak{S}^{\varnothing}) \cong R\Gamma_{\text{rig}}(X).$$

3. Proof of the main result

In this section we prove Theorem 1, stated in the Introduction.

Remark 3. Our setting is the derived category of φ-modules, and it will be implicit that all the quasi-isomorphisms below are compatible with Frobenius when the objects have a non-trivial structure of the φ-module.

Lemma 4. *To prove Theorem 1, it suffices to construct an exact triangle as in the statement where $R\Gamma_{\text{log-crys}}(X_s/S^{\times})$ and its monodromy operator N are replaced by $R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\times})$ and its monodromy operator.*

Proof. By [19, Theorem 3.1.1] and [12, §1.5], there is a map

$$\alpha : R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\times}) \to R\Gamma_{\text{log-crys}}(X_s/S^{\times})$$

which is an isomorphism because we are in the proper and log-smooth case. The map α is obtained by considering that the log-rigid cohomology coincides with the log-convergent one and the latter can be calculated by classical tubes when the immersion is exact. Also, there is a natural map from these classical tubes to the divided-power ones which are used to calculate the log-crystalline cohomology (cf. [3, Proof of proposition 1.9]). In both cases, monodromy operators are defined as the connecting morphism arising from appropriate short exact sequences coming from an embedding system, which can be used for both cohomology theories: we can assume that the embedding system is exact, for instance as in [12, § 5.2]. The log-rigid one is detailed in [12, §5.4] and the log-crystalline one is analogous. □

Remark 5 *(log-rigid comparison).* Note that log-rigid cohomology over $\mathfrak{S}^{\times}$ can alternatively be defined as [10]. The definition of Ertl–Yamada gives cohomology groups isomorphic to those of Große-Klönne [10, Remark 2.4]. In both cases, the monodromy operator is defined starting with a short exact sequence [10, eq. (3.36) and (3.37)] and [12, § 5.4] and there is a natural map between the two.

In fact, in [10] the authors use an embedding system defined over $W[[s]]$, while in [12] it is over $W[s]^\dagger$. The embedding system can be constructed as in [10, Lemma 2.6 and Definition 2.7] both over $W[s]^\dagger$ and over $W[[s]]$, compatibly with the natural map $W[s]^\dagger \to W[[s]]$. From this we get the map of short exact sequences and the monodromy is compatible because of isomorphism between log-rigid cohomology groups over $\mathfrak{S}^\times$.

As a first step, we reformulate the results of [9] entirely in the language of derived categories. Note that the open inclusion $X \setminus X_s \to X$ induces a natural map $\iota : R\Gamma_{\mathrm{rig}}(X, X) \to R\Gamma_{\mathrm{rig}}(X \setminus X_s, X)$. Moreover $R\Gamma_{\mathrm{rig}}(X, X) = R\Gamma_{\mathrm{conv}}(X)$. First we prove

Lemma 6. *There is a canonical isomorphism*

$$R\Gamma_{X_s,\mathrm{rig}}(X) \cong [\iota : R\Gamma_{\mathrm{conv}}(X) \to R\Gamma_{\mathrm{rig}}((X \setminus X_s, X))]$$

where $[-]$ *denotes the homotopy limit.*

In other words, $R\Gamma_{X_s,\mathrm{rig}}(X)$ can be computed without passing to a compactification.

Proof. For our basic landscape to compute rigid cohomology, we fix, as in [9, §4], a simplicial Zariski hypercovering of the form

$$\begin{tikzcd}
X_{s,\bullet} \arrow[r] \arrow[d] & X_\bullet \arrow[r] \arrow[d] & \overline{X}_\bullet \arrow[r] \arrow[d] & \mathscr{P}_\bullet \\
X_s \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \overline{X} & \\
s \arrow[r] & C. & &
\end{tikzcd}$$

Here the simplicial map $\overline{X}_\bullet \to \overline{X}$ is a Zariski affine hypercovering, $\mathscr{P}_\bullet$ is a simplicial formal scheme separated and of finite type over $\mathcal{O}_K$ which is smooth around $X_\bullet$, and which admits a Frobenius $\sigma_\bullet$ lifting that on $\mathcal{O}_K$.[6] Then by definition, we have

$$R\Gamma_{X_s,\mathrm{rig}}(X) = R\Gamma(]\overline{X}_\bullet[_{\mathscr{P}_\bullet}, (j^\dagger_{]X_\bullet[_{\mathscr{P}_\bullet}} \Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}} \to j^\dagger_{]X_\bullet \setminus X_{s,\bullet}[_{\mathscr{P}_\bullet}} \Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}})_s) \tag{3}$$

where $(-)_s$ denotes the total complex of the morphism of complexes, interpreted as the rows of a double complex with $j^\dagger_{]X_\bullet[_{\mathscr{P}_\bullet}} \mathcal{O}_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}}$ in degree $(0,0)$. Here, for $U_\bullet = X_\bullet$ or $X_\bullet \setminus X_{s,\bullet}$, the symbol $j^\dagger_{]U_\bullet[_{\mathscr{P}_\bullet}}$ denotes the functor of overconvergent sections along $]\overline{X}_\bullet \setminus U_\bullet[_{\mathscr{P}_\bullet}$

[6] In [9, §4] the authors also need a compactification $\overline{X}$ over a smooth compactification of C to apply a result of Crew on weight-monodromy in positive characteristic. In the present paper we do not use the result of Crew and hence we do not need such a compactification.

Consider now the following admissible covering of $]\overline{X}_\bullet[_{\mathscr{P}_\bullet}$

$$\{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}\backslash]X_{s,\bullet}[_{\mathscr{P}_\bullet},]X_\bullet[_{\mathscr{P}_\bullet}.\}$$

Note that the two constituent complexes in (3) agree on the former admissible open subset and also on the intersection $(]\overline{X}_\bullet[_{\mathscr{P}_\bullet}\backslash]X_{s,\bullet}[_{\mathscr{P}_\bullet})\cap]X_\bullet[_{\mathscr{P}_\bullet}$. As such the total complex in (3) restricts, on these admissible opens, to the total complex of the identity map, which has trivial cohomology [21, Exercise 1.5.1]. It follows from Zariski descent that

$$\begin{aligned} R\Gamma_{X_s,\mathrm{rig}}(X) &= R\Gamma(]\overline{X}_\bullet[_{\mathscr{P}_\bullet},(j^\dagger_{]X_\bullet[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}} \to j^\dagger_{]X_\bullet\backslash X_{s,\bullet}[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}})_s) \\ &\cong R\Gamma(]\overline{X}_\bullet[_{\mathscr{P}_\bullet}, Ra_{\bullet,*}((j^\dagger_{]X_\bullet[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}})|_{]X_\bullet[_{\mathscr{P}_\bullet}} \\ &\qquad\qquad \to (j^\dagger_{]X_\bullet\backslash X_{s,\bullet}[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}})|_{]X_\bullet[_{\mathscr{P}_\bullet}})_s) \end{aligned}$$

where $a_\bullet :]X_\bullet[_{\mathscr{P}_\bullet}\hookrightarrow]\overline{X}_\bullet[_{\mathscr{P}_\bullet}$ is the inclusion.

Furthermore, it was shown in [9, Proposition 4.1] that the direct image $a_{\bullet,*}$ is exact on the sheaves $\Omega^\bullet_{]X_\bullet[_{\mathscr{P}_\bullet}}$ and $j^\dagger_{]X_\bullet\backslash X_{s,\bullet}[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]X_\bullet[_{\mathscr{P}_\bullet}}$, resulting from the fact that X_s is a divisor. Hence

$$\begin{aligned} R\Gamma_{X_s,\mathrm{rig}}(X) &= R\Gamma(]\overline{X}_\bullet[_{\mathscr{P}_\bullet},(j^\dagger_{]X_\bullet[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}} \to j^\dagger_{]X_\bullet\backslash X_{s,\bullet}[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}})_s) \\ &\cong R\Gamma(]\overline{X}_\bullet[_{\mathscr{P}_\bullet}, Ra_{\bullet,*}((j^\dagger_{]X_\bullet[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}})|_{]X_\bullet[_{\mathscr{P}_\bullet}} \\ &\qquad\qquad \to (j^\dagger_{]X_\bullet\backslash X_{s,\bullet}[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]\overline{X}_\bullet[_{\mathscr{P}_\bullet}})|_{]X_\bullet[_{\mathscr{P}_\bullet}})_s) \\ &\cong R\Gamma(]\overline{X}_\bullet[_{\mathscr{P}_\bullet}, a_{\bullet,*}(\Omega^\bullet_{]X_\bullet[_{\mathscr{P}_\bullet}} \to j^\dagger_{]X_\bullet\backslash X_{s,\bullet}[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]X_\bullet[_{\mathscr{P}_\bullet}})_s) \\ &\cong R\Gamma(]X_\bullet[_{\mathscr{P}_\bullet}, (\Omega^\bullet_{]X_\bullet[_{\mathscr{P}_\bullet}} \to j^\dagger_{]X_\bullet\backslash X_{s,\bullet}[_{\mathscr{P}_\bullet}}\Omega^\bullet_{]X_\bullet[_{\mathscr{P}_\bullet}})_s) \\ &= [R\Gamma_{\mathrm{conv}}(X) \to R\Gamma_{\mathrm{rig}}((X\setminus X_s, X))]. \end{aligned}$$

In the final line we've used the fact that the total complex of a morphism $(A^\bullet \to B^\bullet)_s$ is a representative of its mapping fiber $[A^\bullet \to B^\bullet]$. This is what we wanted to prove. □

Now that we've established that the cohomology $R\Gamma_{X_s,\mathrm{rig}}(X)$ can be computed without a compactification of (a simplicial cover of) X, we are now able to compute $R\Gamma_{X_s,\mathrm{rig}}(X)$ via an alternative covering which replaces the data of a good compactification of X with a covering respecting the log structure on X.

Denote by M the log structure on X associated to the normal crossing divisor X_s. In the following, the scheme X_s will be considered as a log-scheme with log-structure induced by M.

Lemma 7. *There is a canonical isomorphism*

$$R\Gamma_{X_s,\mathrm{rig}}(X) \cong [R\Gamma_{\mathrm{conv}}(X_s) \to R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^\varnothing)].$$

Proof. Denote by M the log structure on X associated to the normal crossing divisor X_s. We also attach to C the log structure N associated to the closed point $s \in C$, so that in particular the morphism $(X, M) \to (C, N)$ is log smooth.

Because C is a smooth curve over k, it admits a smooth lifting $\widetilde{C}$ over $\mathcal{O}_K$ [18, Corollaire III.7.4]. Let $\mathscr{C}$ be the completion of $\widetilde{C}$ along its special fiber C, let $\hat{s}$ be a lift of s to $\mathscr{C}$ and let t be a local coordinate of $\hat{s}$ over $\mathcal{O}_K$. Then $1 \mapsto t$ defines a log structure on $\mathscr{C}$, which we denote by $\mathscr{N}$. Then the maps

$$s^{\times} \to (C, N) \to (\mathscr{C}, \mathscr{N})$$

are exact closed immersions and the latter two are log smooth over $(\operatorname{Spec} k)^{\varnothing}$ and $\mathfrak{S}^{\varnothing}$, respectively.

By [9, Proposition 4.3], we can construct a simplicial étale hypercovering

$$\begin{tikzcd}
(X_\bullet, M_\bullet) \arrow[r, "i_\bullet^{\mathrm{ex}}"] \arrow[d] & (\widetilde{\mathscr{Q}}_\bullet^{\mathrm{ex}}, \widetilde{\mathscr{M}_\bullet}) \arrow[dd] \\
(X, M) \arrow[d] & \\
(C, N) \arrow[r] & (\mathscr{C}, \mathscr{N})
\end{tikzcd}$$

where the log structure $M_\bullet$ on $X_\bullet$ is that induced by M and where $i_\bullet^{\mathrm{ex}}$ is an exact closed immersion into a simplicial log-formal scheme $(\widetilde{\mathscr{Q}}_\bullet^{\mathrm{ex}}, \widetilde{\mathscr{M}_\bullet})$ which is log-smooth over $(\mathscr{C}, \mathscr{N})$ and with $\widetilde{\mathscr{Q}}_\bullet^{\mathrm{ex}}$ separated, smooth, and of finite type over $\mathcal{O}_K$. Moreover $(\widetilde{\mathscr{Q}}_\bullet^{\mathrm{ex}}, \widetilde{\mathscr{M}_\bullet}) \to (\mathscr{C}, \mathscr{N})$ is formally log-smooth and admits a lift of Frobenius.

Remark 8. The virtue of such a hypercovering is that, as we will see, it can be used both to compute rigid cohomology *and* logarithmic rigid cohomology.

If we let $X_{s,\bullet}$ denote the induced étale hypercovering of X_s, we can now apply a pair of results of Shiho [19, Corollary 2.3.9, Proposition 2.4.4] which in conjunction say that we have an identification

$$R\Gamma_{\mathrm{rig}}((X_m \setminus X_{s,m}, X_m)) \cong R\Gamma_{\text{log-conv}}((X_m, M_m)/\mathfrak{S}^{\varnothing}) \ ,$$

because every (X_m, M_m) is a smooth log scheme and its log-structure is trivial on $X_m \setminus X_{s,m}$.

It follows by étale descent on rigid cohomology [8] that

$$R\Gamma_{\mathrm{rig}}((X \setminus X_s, X)) \cong R\Gamma_{\text{log-conv}}((X, M)/\mathfrak{S}^{\varnothing}).$$

It is clear by looking at their representative de Rham complexes that this isomorphism fits into the commutative diagram

$$\begin{tikzcd} R\Gamma_{\mathrm{conv}}(X) \arrow[r] \arrow[d, "="] & R\Gamma_{\text{log-conv}}((X,M)/\mathfrak{S}^{\varnothing}) \arrow[d, "\cong"] \\ R\Gamma_{\mathrm{conv}}(X) \arrow[r] & R\Gamma_{\mathrm{rig}}((X \setminus X_s, X)) \end{tikzcd}$$

so to prove the lemma it suffices to show, by Lemma 6, that

$$[R\Gamma_{\mathrm{conv}}(X) \to R\Gamma_{\text{log-conv}}((X,M)/\mathfrak{S}^{\varnothing})] \cong [R\Gamma_{\mathrm{conv}}(X_s) \to R\Gamma_{\text{log-conv}}((X_s,M_s)/\mathfrak{S}^{\varnothing})].$$

The argument is similar to that of Lemma 6. The former mapping fiber can be written as

$$R\Gamma(]X_\bullet[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}, (\Omega^\bullet_{]X_\bullet[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}} \to \Omega^\bullet_{]X_\bullet[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}}\langle\widetilde{\mathscr{M}_\bullet}\rangle)_s). \tag{4}$$

To compute this, consider the admissible cover of $\mathscr{C}_\eta$ given by the tube of $\{s\}$ in $\mathscr{C}$ and V a strict affinoid neighborhood of $C \setminus \{s\}$ in $\mathscr{C}$. Their inverse image in $]X_\bullet[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}$ is an admissible covering as well; the inverse image of the tube of $\{s\}$ is given by $]X_{s,\bullet}[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}$ and we denote by $V_\bullet$ the inverse image of V. The restrictions to $V_\bullet$ of the two complexes in (4) are the same, and the inclusion $\iota_{\bullet,\eta} :]X_{s,\bullet}[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet} \to]X_\bullet[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}$ is quasi-Stein, again since X_s is a divisor. It thus follows by the same argument as in Lemma 6, using Kiehl's result that $R\iota_{\bullet,\eta*} = \iota_{\bullet,\eta*}$ on coherent sheaves [14, Satz 2.4], that (4) is isomorphic to

$$R\Gamma(]X_{s,\bullet}[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}, (\Omega^\bullet_{]X_{s,\bullet}[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}} \to \Omega^\bullet_{]X_{s,\bullet}[_{\widetilde{\mathcal{Q}}^{\mathrm{ex}}_\bullet}}\langle\widetilde{\mathscr{M}_\bullet}\rangle)_s)$$

But this in turn is isomorphic to

$$[R\Gamma_{\mathrm{conv}}(X_s) \to R\Gamma_{\text{log-conv}}(X_s/\mathfrak{S}^{\varnothing})]$$

as desired. $\square$

We are now in a position to prove the main claim:

Proof of Theorem 1. Due to Lemma 4 we prove the statement with "log-rig" in place of "log-crys" and $R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\times})$ is the log-rigid cohomology complex of Ertl–Yamada (see Remark 5).

By [10, Proposition 3.33 (1)], the associated exact triangle is

$$R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\varnothing}) \to R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\times}) \xrightarrow{N} R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\times})(-1) \to .$$

(The twist $R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\times})(-1)$ arises from the fact that, as expected, $N\varphi = p\varphi N$; see [12, Proposition 5.5]). We thus obtain a quasi-isomorphism

$$R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\varnothing}) \cong [R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\times}) \xrightarrow{N} R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^{\times})(-1)].$$

Plugging this into the exact triangle corresponding to the mapping fiber in Lemma 7 we obtain an exact triangle

$$R\Gamma_{X_s,\mathrm{rig}}(X) \to R\Gamma_{\mathrm{conv}}(X_s) \to \\ [R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^\times) \xrightarrow{N} R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^\times)(-1)] \to .$$

Finally, Poincaré duality [2, Théorème 2.4] (see [6, §2.1] for details on the Frobenius action) provides a canonical isomorphism

$$R\Gamma_{X_s,\mathrm{rig}}(X) \cong R\Gamma_{\mathrm{rig}}(X_s)^*(-n-1)[-2n-2]$$

so after substitution and shifting the triangle we obtain an exact triangle

$$\begin{tikzcd}
 & R\Gamma_{\mathrm{rig}}(X_s)^*(-n-1)[-2n-1] \arrow[dl] & \\
R\Gamma_{\mathrm{rig}}(X_s) \arrow[rr] & & \left[R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^\times) \xrightarrow{N} R\Gamma_{\text{log-rig}}(X_s/\mathfrak{S}^\times)(-1)\right] \arrow[ul]
\end{tikzcd}$$

of φ-modules, as desired. □

Remark 9. Of course a scheme Y in characteristic p can be embedded as a closed subscheme in other ways and these other embeddings suggest further directions for research. For example, Y may be the fiber of a 1-dimensional *arithmetic* family, such as a discrete valuation ring of mixed or equal characteristic.

It may be interesting to consider Y as the special fiber of a scheme X over a complete discrete valuation ring with residue field k and to give meaning to the cohomology

$$R\Gamma_Y(X)$$

with support in the special fiber. It may be possible to extend the proof to this situation if such a cohomology is defined. One could also study a family over a discrete valuation ring of equicharacteristic p, for example when Y is the special fiber of a scheme X over $k[\![t]\!]$. Rigid cohomology over such a Laurent series has a more analytic flavor. It is defined to be a functor

$$X \mapsto H^*_{\mathrm{rig}}(X/\mathscr{E}_K)$$

taking values in graded vector spaces over the Amice ring

$$\mathscr{E}_K := \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \in K[\![t, t^{-1}]\!] \,:\, \sup_i |a_i| < \infty, \lim_{i\to-\infty} a_i = 0 \right\}.$$

To study these objects one can study instead rigid cohomology over the bounded Robba ring $X \mapsto H^*_{\mathrm{rig}}(X/\mathscr{E}^\dagger_K)$, where

$$\mathscr{E}^\dagger_K = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \in K[\![t, t^{-1}]\!] : \sup_i |a_i| < \infty, \exists \eta < 1 \,\mathrm{s.\,t.} \lim_{i\to-\infty} |a_i|\eta^i = 0 \right\}$$

The bounded Robba ring has the additional virtue that it is a Henselian discretely valued field with residue field $k((t))$. This cohomology theory is constructed so that when we base change to $\mathscr{E}_K$ one recovers $\mathscr{E}_K$-valued rigid cohomology (see [16, §2.2] for details). This is a direction for further research.

Data availability

No data was used for the research described in the article.

References

[1] A. Beilinson, On the crystalline period map, Camb. J. Math. 1 (1) (2013) 1–51, https://doi.org/10.4310/CJM.2013.v1.n1.a1.

[2] P. Berthelot, Dualité de Poincaré et formule de Künneth en cohomologie rigide, C. R. Acad. Sci., Sér. 1 Math. 325 (5) (1997) 493–498, https://doi.org/10.1016/S0764-4442(97)88895-7.

[3] P. Berthelot, Finitude et pureté cohomologique en cohomologie rigide, Invent. Math. 128 (2) (1997) 329–377, https://doi.org/10.1007/s002220050143, With an appendix in English by Aise Johan de Jong.

[4] L. Berger, An introduction to the theory of p-adic representations, in: Geometric Aspects of Dwork Theory. Vol. I, II, Walter de Gruyter, Berlin, 2004, pp. 255–292.

[5] F. Binda, M. Gallauer, A. Vezzani, Motivic monodromy and p-adic cohomology theories, arXiv:2306.05099, 2023.

[6] B. Chiarellotto, B. Le Stum, Pentes en cohomologie rigide et F-isocristaux unipotents, Manuscr. Math. 100 (4) (1999) 455–468, https://doi.org/10.1007/s002290050212.

[7] R. Crew, Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve, Ann. Sci. Éc. Norm. Supér. (4) 31 (6) (1998) 717–763, https://doi.org/10.1016/S0012-9593(99)80001-9.

[8] B. Chiarellotto, N. Tsuzuki, Cohomological descent of rigid cohomology for étale coverings, Rend. Semin. Mat. Univ. Padova 109 (2003) 63–215.

[9] B. Chiarellotto, N. Tsuzuki, Clemens-Schmid exact sequence in characteristic p, Math. Ann. 358 (3–4) (2014) 971–1004, https://doi.org/10.1007/s00208-013-0980-8.

[10] V. Ertl, K. Yamada, Rigid analytic reconstruction of Hyodo–Kato theory, https://arxiv.org/abs/1907.10964v5, 2024.

[11] M. Flach, B. Morin, Weil-étale cohomology and zeta-values of proper regular arithmetic schemes, Doc. Math. 23 (2018) 1425–1560.

[12] E. Große-Klönne, Frobenius and monodromy operators in rigid analysis, and Drinfel'd's symmetric space, J. Algebraic Geom. 14 (3) (2005) 391–437, https://doi.org/10.1090/S1056-3911-05-00402-9.

[13] O. Gregory, A. Langer, Overconvergent de Rham–Witt cohomology for semi-stable varieties, Münster J. Math. 13 (2) (2020) 541–571.

[14] R. Kiehl, Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie, Invent. Math. 2 (1967) 256–273, https://doi.org/10.1007/BF01425404.

[15] B. Le Stum, Rigid Cohomology, Cambridge Tracts in Mathematics, vol. 172, Cambridge University Press, Cambridge, 2007.

[16] C. Lazda, A. Pál, Rigid Cohomology over Laurent Series Fields, Algebra and Applications, vol. 21, Springer, Cham, 2016.

[17] J. Nekovář, W.a. Nizioł, Syntomic cohomology and p-adic regulators for varieties over p-adic fields, Algebra Number Theory 10 (8) (2016) 1695–1790, https://doi.org/10.2140/ant.2016.10.1695, With appendices by Laurent Berger and Frédéric Déglise.

[18] Revêtements étales et groupe fondamental (SGA 1), Documents Mathématiques (Paris), vol. 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61.
[19] A. Shiho, Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology, J. Math. Sci. Univ. Tokyo 9 (1) (2002) 1–163.
[20] T. Tsuji, Poincaré duality for logarithmic crystalline cohomology, Compos. Math. 118 (1) (1999) 11–41.
[21] C.A. Weibel, An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
[22] Y.-T. Wu, On the p-adic local invariant cycle theorem, Math. Z. 285 (3–4) (2017) 1125–1139, https://doi.org/10.1007/s00209-016-1741-7.