A note on Grothendieck fundamental group

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Abstract

We give a basic introduction to the Grothendieck fundamental group. We consider only affine schemes and we don't assume the reader familiar with algebraic geometry.

Introduction

Very roughly speaking we can say that algebraic/arithmetic geometry is the study of systems of polynomial equations. Let k be a commutative ring with unit and $f_1, ..., f_m \in k[T_1, ..., T_n]$ be polynomials of n unknowns with coefficient in k. We are interested to study the "solutions" the system

$$\Sigma = \begin{cases} f_1(T_1, ..., T_n) & = 0 \\ \quad \vdots & \vdots \\ f_m(T_1, ..., T_n) & = 0 \end{cases}$$

For any k-algebra R, i.e. $Z_\Sigma(R) := \{a = (a_1, ..., a_n) \in R^n \mid f_i(a_1, ..., a_m) = 0 \ \forall\ i\}$. Two well known particular cases are:

1. if $R = k$ is a field and $n = m = 1$ we are in the setting of Galois theory.
2. if $R = k$ is a field and the f_i are linear we are doing linear algebra.

Another important example is the case $k = \mathbb{C}$, then the set $Z_\Sigma(\mathbb{C})$ can be viewed as a closed sub-set of $\mathbb{C}^n$ with respect to the standard topology. Hence in this case we can use topological methods in order to classify these zero-sets.

In topology there are (at least) two ways to define the fundamental group $\pi_1(S, s)$ of a topological space S. Namely we can view it as the set of loops based on a point s up to homotopy, or as the group of automorphism of the universal cover. This second approach can be made algebraic (i.e. it works for any ring k), as we are going to explain later, and allows to define the Grothendieck (or étale) fundamental group π_1^{et} of a system Σ (or more generally any noetherian connected scheme). This is a pro-finite group and it is isomorphic to the pro-finite completion of $\pi_1(Z_\Sigma(\mathbb{C}), s)$ when $k = \mathbb{C}$. This follows by some important differences between the topological and the algebraic setting.

In fact in the algebraic setting one cannot use the topological definition of a cover. This lead to the notion of étale morphism which is purely algebraic and corresponds to local isomorphisms when we are dealing with complex schemes.
There is also another problem to overcome. The universal étale cover rarely exists. The solution is to consider finite approximations. It follows that the étale fundamental group is a projective limit of finite groups.

1 Affine schemes

Let $f_1, ..., f_m \in k[T_1, ..., T_n]$ and consider the system of equations

$$\Sigma = \begin{cases} f_1(T_1, ..., T_n) & = 0 \\ \quad\vdots & \vdots \\ f_m(T_1, ..., T_n) & = 0 \end{cases}$$

Let $Z_\Sigma(R) := \{a = (a_1, ..., a_n) \in R^n \mid f_i(a_1, ..., a_m) = 0 \ \forall \ i\}$ be the set of solutions of the system with values in R.

In order to have a better understanding of $Z_\Sigma(R)$ for any R we will use commutative algebra and study the $A = k[T_1, ..., T_n]/I$ associated to the system Σ, where $I = (f_1, ..., f_m)$. This is justified by the following

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Alg}_k}(A, R) &= \mathrm{Hom}_{\mathsf{Alg}_k}\left(\frac{k[T_1, ..., T_n]}{(f_1, ..., f_m)}, R\right) \\ &= \{\phi : k[T_1, ..., T_n] \to R \mid \phi(f_i) = 0 \ \forall \ i\} \\ &= \{(a_1 = \phi(T_1), ..., a_n = \phi(T_n)) \mid \ f_i(a_1, ..., a_n) = 0 \ \forall \ i\} \\ &= Z_\Sigma(R) \ . \end{aligned}$$

Note that given a ring morphism $f : A \to B$ we have a natural transformation $f^* : \mathrm{Hom}_{\mathsf{Alg}_k}(B, -) \to \mathrm{Hom}_{\mathsf{Alg}_k}(A, -)$. Explicitly for any ring R we have $f^*_R : \mathrm{Hom}_{\mathsf{Alg}_k}(B, R) \to \mathrm{Hom}_{\mathsf{Alg}_k}(A, R)$, $f^*_R(g) = g \circ f$.

Example 1.1. i) Let k be a field. Then for any k-algebra R, $\mathrm{Hom}_{\mathsf{Alg}_k}(k, R) = \{pt\}$. In fact there is only one k-linear morphism sending $1 \mapsto 1$.

ii) Let $A = k[T_1, ..., T_n]$ and R be an k-algebra. Then $\mathrm{Hom}(k[T_1, ..., T_n], R) = R^n$. Hence this ring corresponds to the n-dimensional affine space over k.

iii) Let $A = k[T, T^{-1}]$ be the ring of Laurent polynomials. Then $\mathrm{Hom}_{\mathsf{Alg}_k}(A, R) = R^*$ is the set of invertible elements in R. This can be thought as the affine line (over k) without the origin.

iv) Let A be a $\mathbb{C}$-algebra of the following type $\mathbb{C}[T_1, ..., T_n]/(f_1, ..., f_m)$ then the set of $\mathbb{C}$-points (or solutions over $\mathbb{C}$) $\mathrm{Hom}_{\mathsf{Alg}_k}(A, \mathbb{C})$ is naturally a closed subset of $\mathbb{C}^n$ w.r.t. the standard topology.

2 The topological fundamental group

Fix a (connected) topological space S. We assume that all the topological spaces in this section are path-connected and locally simply connected. These assumptions on S imply the existence of the universal cover.

Definition 2.1. We say that a continuos and surjective map $\phi : X \to S$ is a *cover* of S (or a *covering space*) if for any point $s \in S$ there exists U, path-connected open neighborhood of s, such that the restriction

$$\phi|_V : V \longrightarrow U$$

is an homeomorphism for any V path-connected component of $\phi^{-1}(U) \subset X$.

Example 2.2. i) Let $S = \mathbb{C}^* = \mathbb{C}\backslash 0$. Then the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$, $\exp(x) := \sum_n x^n/n!$, is a cover of $\mathbb{C}^*$. Note that $\# \exp^{-1}(s) = \infty$ for any $s \in \mathbb{C}^*$.

ii) Let $S = \mathbb{C}^*$ and fix $n \in \mathbb{Z}$. Then we can define the following cover of S $\mu_n : \mathbb{C}^* \to \mathbb{C}^*$, $\phi(x) = x^n$. This cover is finite of degree n: i.e. $\#\mu_n^{-1}(s) = n$ for any $s \in \mathbb{C}^*$.

We define the category Cov_S of covers of S in the following way: an object of Cov_S is a cover $\phi : X \to S$; a morphism f from $\phi : X \to S$ to $\psi : Y \to S$ is a continuous map $f : X \to Y$ such that $\psi \circ f = \phi$.

Remark 2.3. Fix a cover $\phi : X \to S$ and a point $s \in S$.

i) According to the previous definition the group $\mathrm{Aut}_{\mathsf{Cov}_S}(\phi)$ is the set of homeomorphisms $f : X \to X$ such that $\phi \circ f = \phi$. This group acts on the fiber $\phi^{-1}(s)$ by

$$\mathrm{Aut}_{\mathsf{Cov}_S}(\phi) \times \phi^{-1}s \to \phi^{-1}(s) \quad (f, x) \mapsto f(x)$$

ii) The previous action is transitive if and only if $\phi_*(\pi_1(X, x)) \subset \pi_1(S, s)$ is a normal sub-group. In this case ϕ is called a Galois cover.

iii) Also the fundamental group $\pi_1(S, s)$ acts on $\phi^{-1}(s)$ by lifting paths

$$\pi_1(S, s) \times \phi^{-1}s \to \phi^{-1}(s) \quad ([\gamma], x) \mapsto \gamma_x(x)$$

where γ_x is a lifting of γ starting from x.
The stabilizer of $x \in \phi^{-1}(s)$ is isomorphic to $\phi_*(\pi_1(X, x))$.

Theorem 2.4. *The functor* $F : \mathsf{Cov}_S \to \mathsf{Set}$*,* $F(X \xrightarrow{\phi} S) := \phi^{-1}(s)$ *is representable: i.e. there exists a (universal) cover* $\phi^u : X^u \to S$ *such that*

$$\mathrm{Hom}_{\mathsf{Cov}_S}(\phi^u, -) \cong F(-)$$

Moreover F *induces an equivalence between the category* Cov_S *and the category of* $\pi_1(S, s)$*-sets.*

Remark 2.5. i) The universal cover can be characterized by the fact that X is simply connected. The existence of such a cover depends on the topological assumptions given at the beginning of the section.

ii) An immediate corollary of the theorem is the canonical isomorphism

$$\pi_1(S, s) \cong \mathrm{Aut}_{\mathsf{Cov}(S)}(\phi^u)$$

hence the topological fundamental group can be defined via covers.

Example 2.6. In the case $S = \mathbb{C}^*$ the exponential map is the universal cover. An automorphism of the cover $\exp : \mathbb{C} \to \mathbb{C}^*$ is of the form $x \mapsto x + 2\pi i n$ for some $n \in \mathbb{Z}$. Hence $\pi_1(\mathbb{C}^*, s_0) \cong 2\pi i\mathbb{Z}$.

3 Grothendieck fundamental group

3.1 étale covers

Definition 3.1. Let A be a ring (noetherian, commutative, with unit). A ring morphism $f : A \to B$ is an étale if it is of finite type, flat and unramified. If it also finite we call it an étale cover.

This means that an étale cover of A is an A-algebra B which is a finitely generated A-module and for any $\mathfrak{m} \subset A$ maximal ideal the fiber $B \otimes A/\mathfrak{m}$ is a separable algebra over the field $A/\mathfrak{m}$. Recall that a separable algebra over a field k is isomorphic to a direct sum of finite separable extensions of k.
On the geometric side we have a map of functors $\phi = f^* : \mathrm{Hom}_{\mathsf{Alg}}(B, -) \to \mathrm{Hom}_{\mathsf{Alg}}(A, -)$ such that for any element (or k-point) $s \in \mathrm{Hom}_{\mathsf{Alg}}(A, k)$ the fiber $\phi^{-1}(s) \subset \mathrm{Hom}_{\mathsf{Alg}}(B, k)$ is a finite set of distinct point as in the topological case.

The basic example of an étale cover is the following.

Example 3.2. Let $A = k[T]$ where k is an algebraically closed field. Consider $B = k[T, S]/(f)$ and $\phi : A \to B$ defined by $\phi(T) = T \mod (f)$. Then ϕ is an étale cover of A if and only if for any $(a, b) \in \mathrm{Hom}_{\mathsf{Alg}}(B, k)$ we have $(\partial f / \partial S)(a, b) \neq 0$.

In particular it is easy to check that we have an étale cover for $f = S - T^2$ (this is in fact an isomorphism), while for $f = S^2 - T$ there is a pathological point in $(0, 0)$.

A key point for the construction of the étale fundamental group is the existence of Galois covers. Let A be a ring and $f : A \to B$ an étale cover, it is a Galois cover there exists a finite group G acting (on the left) faithfully on B such that

1. The rings of G-invarinats $B^G = \{g \in G | gb = b\}$ is isomorphic to A.

2. $f : A \to B$ is the canonical inclusion $B^G \to B$.

In this case the group of automorphism of the cover is G. If we start with G acting on B than the canonical map $B^G \to B$ is an étale cover if the all inertia groups of the action are trivial.

Example 3.3. Let $G = \mathbb{Z}/n\mathbb{Z}$ and $B = k[T, T^{-1}, S]/(S^n - T)$. Consider the following action: $[m] \cdot T = T$, $[m] \cdot S = S^m$, for $m \in \mathbb{Z}$. Then it easy to check that $B^G = k[T, T^{-1}]$ and that the canonical map $\iota : k[T, T^{-1}] \to k[T, T^{-1}, S]/(S^n - T)$ is étale. Moreover note that there is another way to write the same cover, namely

$$\mu_n : k[T, T^{-1}] \to k[T, T^{-1}] \quad \mu_n(T) = T^n$$

in fact it is easy to check that there is a ring ismorphism $\theta : k[T, T^{-1}, S]/(S^n - T) \to k[T, T^{-1}]$ such that $\theta \circ \iota = \mu_n$.

3.2 The main result

Let A be a ring (e.g. $A = k[T_1, ..., T_n]/I$) and fix $s \in \mathrm{Hom}_{\mathsf{Alg}}(A, \Omega)$ where Ω is an algebraically closed field. For any étale $f : A \to B$ cover we can consider the set of points of $\mathrm{Hom}_{\mathsf{Alg}}(B, \Omega)$ lying over s, i.e.

$$F(B) := \{t : B \to \Omega \,| t \circ f = s\}$$

This association induces a functor from the category of étale covers of A to the category of sets. Now we are ready to state the main result of these notes.

Theorem 3.4. *There exists a projective limit of Galois cover $(f_\alpha : A \to B_\alpha)$ such that for any étale cover $f : A \to B$*

$$F(B) \cong \operatorname*{colim}_\alpha \operatorname{Hom}_{\mathsf{Alg}_A}(B, B_\alpha)$$

(i.e. F is a pro-representable functor). Moreover we can define the pro-finite group $\pi_1^{\text{et}}(A, s) := \lim_\alpha \operatorname{Aut}(B_\alpha) = \lim_\alpha F(B_\alpha)$ and F induces an equivalence of categories

$$F : \{\textit{étale covers of } A\} \to \{\pi_1^{\text{et}}(A, s)\textit{-sets}\} .$$

Proof. See [sga71, Exp. V, §5 and 7]. □

Example 3.5. We already know that the (topological) fundamental group of $\mathbb{C}^*$ is isomorphic to $\mathbb{Z}$. In the algebraic setting $\mathbb{C}^*$ corresponds[1] to the ring $A = \mathbb{Q}[T, T^{-1}]$. The following morphism

$$(-)^n : \mathbb{Q}[T, T^{-1}] \to B_n = \mathbb{Q}[T, T^{-1}] \quad T \mapsto T^n$$

is an étale cover, moreover it is a Galois cover with group $\mathbb{Z}/n\mathbb{Z}$. Hence if we take the projective limit $\lim_n \operatorname{Aut}(B_n) = \lim_n \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}$ we obtain the pro-finite completion of $\mathbb{Z}$.

Example 3.6. Let $A = \mathbb{R}$, then it is easy to check that $\mathbb{R} \to \mathbb{C}$ is an étale cover with group $\mathbb{Z}/2\mathbb{Z}$ (the action is the complex conjugation!). Moreover this the universal étale cover of $\mathbb{R}$ and $\pi_1^{\text{et}}(\mathbb{R}, \mathbb{C}) = \operatorname{Aut}(\mathbb{R} \to \mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$. We remark that this is a very special situation. In fact we can hope to find the universal étale cover only in case π_1^{et} is a finite group.

All this theory can be generalized to the case of locally noetherian affine schemes. Unfortunately we can compute the étale fundamental group only in few cases, even for affine schemes. As final remarks we mention the following important results

1. (Comparison with the topological fundamental group) Let A be a $\mathbb{C}$-algebra of finite type (i.e. $A = \mathbb{C}[T_1, ..., T_n]/I$) and let S be the set of $\mathbb{C}$-points $\operatorname{Hom}_{\mathsf{Alg}}(A, \mathbb{C})$ endowed with the standard topology. Then there is a canonical isomorphism of pro-finite groups

$$\widehat{\pi}_1(S, s) \cong \pi_1^{\text{et}}(A, s)$$

where $\widehat{\pi}_1(S, s)$ is the completion of the (topological) fundamental group $\pi_1(S, s)$ w.r.t. the topology of finite index sub-groups. (See [sga71, XII.5.2]).

[1]Indeed we could consider any other field in place of $\mathbb{Q}$, but the story changes a lot in characteristic p.

2. (Galois theory) If k is a field and $\bar{k}$ is its algebraic closure, then $\pi_1^{\text{et}}(k, \bar{k}) = \text{Gal}(\bar{k}/k)$ is the absolute Galois group of k.

3. If A is an algebra over a field k, then there is an exact sequence

$$1 \to \pi_1^{\text{et}}(A \otimes_k \bar{k}, \bar{k}) \to \pi_1^{\text{et}}(A, \bar{k}) \to \text{Gal}(\bar{k}/k) \to 1$$

   Thus in general the étale fundamental group carries both a geometric and an arithmetic information.

References

[sga71] *Revêtements étales et groupe fondamental.* Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. 224.