AN INTRODUCTION TO FORMAL GEOMETRY

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Abstract. The first aim of this 5th talk (of about 1h20) is to give an account on some basic results of formal geometry, including Grothendieck's Comparison and Existence Theorems.

We will consider only noetherian formal schemes, and cover the content of Grothendieck's Bourbaki seminar on the subject [3, Exp. 182, §§ 1-3]. We recall the basic definitions following the notes of Illusie [5]. We also recall the main results (without proof) about completions and Mittag-Leffler conditions.

Concerning Grothendieck's Comparison and Existence Theorems, some ideas of their proofs will be given, under some projectivity (instead of mere properness) assumption. (The proof of the Existence Theorem is quite close to the one of GAGA Existence Theorem.)

This talk will include a discussion of: (i) the formal completion of some noetherian scheme along a closed subscheme; (ii) the notion of "algebraic formal germ" as in [1, 4.2], and of its higher dimensional variant in [2, Proposition A.1]; (iii) the "formal exponential map" associated to an algebraic group G over a field k of characteristic zero [1, 2.1.2].

Grazie mille to the organizers of this workshop!

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1. Basic definitions

1.1. **The objects of algebraic geometry.** Roughly speaking we can say that the object of arithmetic algebraic geometry is the study of systems of plynomial equations

$$X = \begin{cases} f_1(x_1, ..., x_n) = 0 \\ \vdots \quad \vdots \\ f_m(x_1, ..., x_n) = 0 \end{cases} ,$$

where $f_i \in k[x_1, ..., x_n]$ are plynomials with coefficients in a field (or a ring) k. For instance one can take $k = \mathbb{Q}$ and ask for solutions in $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. In general we can denote by $X(R)$ the set of R-solutions of the system X and we can easily check that

$$X(R) = \mathrm{Hom}_{\mathrm{Alg}_k}(\frac{k[x_1, ..., x_n]}{(f_1, ..., f_m)}, R) \subset R^n \ .$$

We can even consider system of homogeneus polynomials so that the set of solutions is contained in some projective space. Anyhow, at least locally, we can always study the functor

$$X(-) = \mathrm{Hom}_{\mathrm{Alg}_k}(\frac{k[x_1, ..., x_n]}{(f_1, ..., f_m)}, -)$$

which is completely determined by the ring $\frac{k[x_1,...,x_n]}{(f_1,...,f_m)}$. For instance

$$\mathrm{Hom}_{\mathrm{Alg}_k}(k[x_1, ..., x_n], k) = k^n \quad \mathrm{Hom}_{\mathrm{Alg}_k}(\frac{k[x, y]}{(xy - 1)}, k) = k^* \ .$$

We could introduce $\mathbb{C}$-anlytic spaces in a similar way by replacing $k[x_1, ..., x_n]$ with the set of holomorphic functions on a polydisk (or some open of $\mathbb{C}^n$).

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1.2. **Local.** An *adic noetherian ring* is a noetherian topological ring A which is separated and complete for the I-adic topology, where I is an ideal of A. In other words

$$A = \lim_{n\geq 0} A/I^{n+1} \ .$$

With such a ring is associated a topologically ringed space[1]

$$\mathrm{Spf}(A) = \operatorname*{colim}_{n\geq 0} \mathrm{Spec}(A/I^{n+1}) \quad \text{(as top. ringed spaces)}$$

called the *formal spectrum* of A. Note that the underling topological space is that of $\mathrm{Spec}(A/I)$ and the structural sheaf is given by the following limit

$$\mathcal{O}_{\mathrm{Spf}(A)} = \lim_{n\geq 0} \mathcal{O}_{\mathrm{Spec}(A/I^{n+1})} \ .$$

An *affine noetherian formal scheme* is a topologically ringed space isomorphic to $\mathrm{Spf}(A)$ for some A as above.

Similarly to previous paragraph let $f_i \in k[[x_1, ..., x_n]]$ (this is the $(x_1, ..., x_n)$-adic completion fo $k[x_1, ..., x_n]$) then $A = k[[x_1, ..., x_n]]/(f_1, ..., f_m)$ is an adic ring and

$$\mathrm{Spf}(A)(R) = \mathrm{Hom}^c_{\mathrm{Alg}_k}(k[[x_1, ..., x_n]]/(f_1, ..., f_m), R) = \operatorname*{colim}_i \mathrm{Hom}_{\mathrm{Alg}_k}(\frac{k[x_\bullet]}{(f_\bullet) + (x_\bullet)^{i+1}}, R)$$

it is just the union of R-solution of a family of system of plolynomials. For instance we have

$$\mathrm{Hom}^c_{\mathrm{Alg}_{\mathbb{Z}}}(\mathbb{Z}[[x_1, ..., x_n]], R) = \mathrm{Nil}(R)^n \quad \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{Z}}}(\frac{\mathbb{Z}[x, y]}{(xy-1)}, R) = 1 + \mathrm{Nil}(R) \ .$$

1.2.1. *Examples.*

(1) A is any ring, $I = (0)$, then $\mathrm{Spf}(A) = \mathrm{Spec}(A)$ is just the affine scheme corresponding to A.
(2) $A = \mathbb{Z}_p$ et $I = p\mathbb{Z}_p$. In this case $|\mathrm{Spf}(\mathbb{Z}_p)| = |\mathrm{Spec}(\mathbb{F}_p)|$ is the $\mathbb{F}_p$-point and the structural sheaf is $\mathbb{Z}_p$
(3) $A = K[[t]]$, $I = tK[[t]]$.

1.3. **Global.** A *locally noetherian formal scheme* is a topologically ringed space locally isomorphic to an affine noetherian formal scheme. Locally noetherian formal schemes form a category whose morphisms are those of locally topologically ringed spaces (i.e. on the structural sheaves they are *local* and *continuous*).

As for schemes we have

$$\mathrm{Hom}(\mathscr{X}, \mathrm{Spf}(A)) = \mathrm{Hom}^c(A, \Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \ .$$

1.4. **The formal completion along a closed subscheme.** Locally noetherian schemes usually appears as colimits of increasing chains of nilpotent thickenings. Let X be a locally noetherian scheme, $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal defining the closed subscheme $Z = V(\mathcal{I})$. Consider the inductive system of (locally noetherian) schemes $X_n = \mathrm{Spec}(\mathcal{O}_X/\mathcal{I}^{n+1})$. This system satisfies :

(1) $X_0 = Z$ is a locally noetherian scheme.
(2) the maps $|X_n| \to |X_{n+1}|$ are homeomorphisms and $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$ are surjections.
(3) for $m \geq n$, $\ker(\mathcal{O}_{X_n} \to \mathcal{O}_{X_m}) = (\ker(\mathcal{O}_{X_n} \to \mathcal{O}_{X_0}))^{m+1}$.
(4) $\ker(\mathcal{O}_{X_1} \to \mathcal{O}_{X_0})$ is a coherent $\mathcal{O}_{X_0}$-module.

The above conditions assure that $\widehat{X} = X_{/Z} := (|X_0|, \lim_n \mathcal{O}_{X_n})$ is a locally noetherian formal scheme. In fact we have $\widehat{X} = \mathrm{colim}_n X_n$ where we do the colimit in the category of formal schemes.

[1]Ringed spaces whose structural sheaf as values in topological rings.

1.5. **Formal germ.** Let $\hat{X} = X_{/Z}$ the completion of a scheme X along a closed subscheme as in the previous number. We assume all the objects are defined over a field k. A *smooth formal germ of X through Z* is a closed formal subscheme $\mathcal{V} \subset \widehat{X}$ such that

(1) $\mathcal{V}$ is formally smooth over k of pure dimension d;
(2) $\mathcal{V}$ contains Z and they have the same underlying topological spaces.

For instance one can take $X = \mathbb{A}^2_k$, $Z = (0,0)$, $\widehat{X} = \mathrm{Spf}(k[[x,y]])$, $\mathcal{V} = \mathrm{Spf}(k[[x]])$ which is algebrizable. If we take $\mathcal{V} = \mathrm{Spf}(k[[x,y]]/(y - \exp(x)))$ we get a non algebrizable formal germ.

1.6. **Group schemes.** Let k be a field of characteristic zero and let G/k be a smooth algebraic group scheme. We denote by $\mathfrak{g}$ its Lie algebra, i.e. the vector group whose functor of points is

$$A \mapsto \ker(G(A[\epsilon]) \xrightarrow{\epsilon \mapsto 0} G(A)) \ .$$

We denote by $\widehat{G}$ (resp. $\widehat{\mathfrak{g}}$) the formal completion of G (resp. $\mathfrak{g}$) along the unit section. Then there exists a canonical isomorphism of formal schemes[2] over k

$$\widehat{\exp_G} : \widehat{\mathfrak{g}} \longrightarrow \widehat{G}$$

such that

(1) the differential in 0 is the identity;
(2) for any homomorphism $\iota : \widehat{\mathbb{G}_{\mathrm{a}}} \to \widehat{\mathfrak{g}}$ the composition $\widehat{\exp_G} \circ \iota$ is a morphism of formal group schemes.

This map is called the *formal exponential map of G*. This map is constructed via the Campbell-Hausdorff formula.

1.6.1. *The trivial case.* For $G = \mathbb{G}_{\mathrm{a}}$ the formal exponential $\widehat{\exp_G}$ is just the identity on the formal additive group.

1.6.2. *The multiplicative group.* For $G = \mathbb{G}_{\mathrm{m}}$ we can identify $\mathfrak{g} = \mathbb{G}_{\mathrm{a}}$ so that for any k-algebra

$$\widehat{\mathfrak{g}}(A) = \mathrm{Nil}(A) \qquad , \quad \widehat{G}(A) = 1 + \mathrm{Nil}(A) \ ,$$

where $\mathrm{Nil}(A)$ is the ideal formed by nilpotent elements in A. In this case the exponential formal series makes sense and it is exactly what we need.

1.6.3. *The general linear group.* For $G = \mathrm{GL}_n$ we can identify $\mathfrak{g} = \mathrm{M}_n$ and for any k-algebra A we have (quite easily)

$$\widehat{\mathfrak{g}}(A) \cong \mathrm{M}_n(\mathrm{Nil}(A)) \quad , \quad \widehat{G}(A) = I + \mathrm{M}_n(\mathrm{Nil}(A))$$

so that the exponential of matrices gives the bijection, but this is not a group homomorphism (unless $n = 1$).

2. The Grothendieck comparison theorem and its corollaries

2.1. **Setting.** Let $f : X \to Y$ be a morphism of locally noetherian schemes, let $Y' \subset Y$ be a closed subset and $X' = f^{-1}(Y')$ so that we can form a morphism f_n, and its limit $\hat{f}$, sitting in the following commutative squares of (formal) schemes

$$\begin{array}{ccc} X_n & \xrightarrow{i_n} & X \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f} \\ Y_n & \xrightarrow{j_n} & Y \end{array} \qquad\qquad \begin{array}{ccc} \widehat{X} & \xrightarrow{i} & X \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widehat{Y} & \xrightarrow{j} & Y \end{array}$$

[2]Not of formal group schemes!

where $\widehat{X} = X_{/X'}$, $\widehat{Y} = Y_{/Y'}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, then the above diagram induces a canonical base change map

$$\gamma : j^* R^q f_* \mathcal{F} \to R^q \hat{f}_*(i^* \mathcal{F})$$

which is $\mathcal{O}_{\widehat{Y}}$-linear.

2.2. **Mittag-Leffler.** *We say that an inverse system of abelian groups* $(A_n, \phi_{n',n})$ *satisfies the ML condition if for each* n *there is an* $n_0 \geq n$ *s.t. for all* $n', n'' \geq n_0$ *we have* $\phi_{n',n}(A_{n'}) = \phi_{n'',n}(A_{n''})$. *This is a sufficient condition for the vanishing* $R^q \lim_n A_n = 0$, *for all* $q > 0$.[3]

2.3. **Artin-Rees Lemma.** *Let* A *be a noetherian ring,* $I \subset A$ *an ideal and* M *a finitely generated* A*-module. For all* $M' \subset M$ A*-submodule there exists* $n_0 > 0$ *such that*

$$\forall\ n \geq n_0, \quad M' \cap I^n M = I^{n-n_0}(M' \cap I_0^n M)\ .$$

In particular we have

1. *The* I*-adic topology on* M *induces the* I*-adic topology on* M'.
2. *The* I*-adic completion is exact on the category of finitely generated* A*-modules.*
3. *There is a canonical isomorphism* $\widehat{A} \otimes_A M \cong \widehat{M}$.

2.4. **The comparison Theorem.** [4] *Notation as in § 2.1. We assume further that*

- $\mathcal{F}$ *is coherent. This implies* $i^*\mathcal{F} = \lim \mathcal{F} \otimes \mathcal{O}_{X_n} =: \widehat{\mathcal{F}}$.
- f *is of finite type and the support of* $\mathcal{F}$ *is proper over* Y*. This implies that* $j^* R^q f_* \mathcal{F}$ *is coherent and we can write* $j^* R^q f_* \mathcal{F} = \widehat{R^q f_* \mathcal{F}}$.

Then the base change map can be written as follows

$$\gamma : \widehat{R^q f_* \mathcal{F}} \longrightarrow R^q \hat{f}_* \widehat{\mathcal{F}}$$

and it is a topological isomorphism of $\mathcal{O}_{\widehat{Y}}$*-modules (for all* q*).*

Sketch of the proof. We can reduce to the case where f is proper by restriction to the support of $\mathcal{F}$. Moreover it is sufficient to prove the theorem for $Y = \operatorname{Spec}(A)$ affine with A noetherian and $Y' = \operatorname{Spec}(A/I)$.

In this setting $\widehat{Y} = \operatorname{Spf}(\widehat{A})$, with $\widehat{A} = \lim_n A/I^{n+1}$, and we can write[5]

$$\widehat{R^q f_* \mathcal{F}} = \lim_n H^q(X, \mathcal{F})/I^{n+1} \qquad R^q \hat{f}_* \widehat{\mathcal{F}} = H^q(\widehat{X}, \widehat{\mathcal{F}})\ ,$$

which are $\widehat{A}$-modules.

Then we prove the following isomorphisms of $\widehat{A}$-modules

$$H^q(\widehat{X}, \widehat{\mathcal{F}}) \xrightarrow[\cong]{a} \lim_n H^q(X, \mathcal{F}/I^{n+1}) \xleftarrow[\cong]{b} \lim_n H^q(X, \mathcal{F})/I^{n+1}\ ,$$

this is enough since $\gamma = a^{-1} \circ b$.

For both isomorphisms we need to deal with the (inverse) limit functor.

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[3] In particular given an exact sequence of inverse systems of abelian groups

$$0 \to (A_n) \to (B_n) \to (C_n) \to 0$$

we get a long exact sequence of inverse limits

$$0 \to \lim A_n \to \lim B_n \to \lim C_n \to R^1 \lim A_n \to \ldots$$

and the ML condition assures the vanishing of $R^1 \lim A_n$ in order to have a short exact sequence.

[4] The original proof of Grothendieck is not published. It should be in the spirit of [4, Ch. III, Theorem 11.1] where f is projective and Y a point. The general case, following an argument of Serre, is proven in [**?**, 4.1.7-8].

[5] By standard computation of higher direct images with values in affine schemes.

(a) Continuity. We start be recalling the following natural isomorphism of (derived) functors

$$R\Gamma(\widehat{X}, R\lim_n \mathcal{F}_n) = R\lim_n \mathbb{R}\Gamma(\widehat{X}, \mathcal{F}_n) \ .$$

We can easily prove that $\Gamma(U, \mathcal{F}_n)$ satisfies ML for any affine U[6], so that $R\lim_n \mathcal{F}_n = \lim_n \mathcal{F}_n$. Then we can consider a spectral sequence

$$R^p \lim_n H^q(\widehat{X}, \mathcal{F}_n) \ \Rightarrow H^{p+q}(\widehat{X}, \widehat{\mathcal{F}}) \ .$$

To get the claim we show that the inverse system $H^q(\widehat{X}, \mathcal{F}_n)$ satisfies ML. This is more difficult and involves the Artin-Rees lemma.

(b) Completion. For this part we follow Hartshorne and we restrict to the case of f projective. Thus we can $X = \mathbb{P}^N_A$ is a projective space. By an explicit calculation[7] we can show (b) when $\mathcal{F}$ is (a finite direct sum of sheaves of the form) $\mathcal{O}(d)$. Next we prove the result for any coherent $\mathcal{F}$ by descending induction on q. By standard vanishing we have $H^q(\mathbb{P}^N_?, \mathrm{coh}) = 0$ if $q > N$. So we assume the comparison true for any sheaf and in degree $q+1$. Let us consider a devissage

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0 \ ,$$

where $\mathcal{E}$ is a finite direct sum of sheaves of the form $\mathcal{O}(d)$. Since tensoring with A/I^{n+1} is not left exact we get two short exact sequences

$$\mathcal{T}_n \hookrightarrow \mathcal{E}_n \twoheadrightarrow \mathcal{F}_n \qquad \mathcal{S}_n \hookrightarrow \mathcal{R}_n \twoheadrightarrow \mathcal{T}_n \ .$$

Here again we need ML and the 5 lemma to reduce the statement to the following claim

$$\forall q \geq 0, \qquad \lim_n H^q(\widehat{X}, \mathcal{S}_n) = 0 \ .$$

The above vanishing follows from the Artin-Rees Lemma. □

3. Corollaries of the comparison Theorem

We keep the notation of §2.1.

3.1. **Ext and completion.** *Assume* $Y = \mathrm{Spec}(A)$, *A noetherian,* $Y' = \mathrm{Spec}(A/I)$, *f morphism of finite type. Let F, G be coherent sheaves on X whose supports have an intersection which is proper over Y. Then* $\mathrm{Ext}^r(F, G)$ *is an A-module of finite type and there is an isomorphism*

$$\widehat{\mathrm{Ext}^r(F, G)} \xrightarrow{\cong} \mathrm{Ext}^r(\hat{F}, \hat{G}) \ .$$

3.2. **Theorem on formal functions.** f is proper, $Y' = y$ is a closed point. *Then for all $q \in \mathbb{Z}$, the stalk $(R^q f_* F)_y$ is an $\mathcal{O}_{Y,y}$-module of finite type and the following map is an isomorphism*

$$(\widehat{R^q f_* F})_y \xrightarrow{\cong} \lim_n H^q(X_y, F_n) \ .$$

3.3. **Zariski's connectedness theorem.** *Let f be proper and* $Y' = \mathrm{Spec}(f_* \mathcal{O}_X)$ *which is a finite scheme over Y sitting in the following factorisation*

$$X \xrightarrow{p} Y' \xrightarrow{g} Y \quad (\textit{Stein factorisation})$$

with p proper and g finite. Then $p_* \mathcal{O}_X = \mathcal{O}_{Y'}$ and the fibres of p are connected and nonempty.

[6] By usual coherent cohomology vanishing.
[7] The invertible sheaf cohomology of a projective space.

3.4. **Zariski's Main Theorem.** *Let f be compactifiable. If f is quasi-finite*[8], *then f can be factored as*

$$X \xrightarrow{j} Z \xrightarrow{g} Y$$

where j is an open immersion and g is a finite morphism.

4. Algebraization problem

4.1. **The problem.** In this section A is an adic noetherian ring and I is an ideal of definition. Let $Y = \mathrm{Spec}(A)$, $Y_n = \mathrm{Spec}(A/I^{n+1})$ and $\widehat{Y} = \mathrm{Spf}(A) = \mathrm{colim}_n Y_n$. Let $\mathcal{X}$ be an adic noetherian formal scheme over $\widehat{Y}$, i.e.

$$\mathcal{X} = \operatorname*{colim}_n X_n \ , \quad X_n := \mathcal{X} \times_{\widehat{Y}} Y_n \ .$$

Then we say that $\mathcal{X}$ is *algebraizable* if it is the I-adic completion of a locally noetherian scheme over Y. If such a scheme exists we may ask about conditions for its uniqueness.

4.2. **Existence Theorem.** *Notation as in § 2.1. Assume that X is noetherian, f of finite type and $Y = \mathrm{Spec}(A)$ affine with. We also assume that A is an adic ring for the ideal I and $Y' = \mathrm{Spec}(A/I)$, so that $\widehat{Y} = \mathrm{Spf}(A)$. Then the I-adic completion induces an equivalence between the following categories :*

(1) *coherent sheaves on X whose support is proper over Y.*
(2) *coherent sheaves on $\widehat{X}$ whose support is proper over $\widehat{Y}$.*

Sketch of the proof. We get fully faithfulness by § 3.1. We will outline the strategy to prove it is essentially surjective.

a) The projective case. If f is projective and E a coherent sheaf on $\widehat{X}$ we can find (with some work) a presentation

$$\mathcal{O}_{\hat{X}}(-m_1)^{r_1} \xrightarrow{u} \mathcal{O}_{\hat{X}}(-m_0)^{r_0} \to E \to 0$$

where $m_i, r_i \geq 0$ and $\mathcal{O}_{\hat{X}}(n) = \mathcal{O}_{\hat{X}} \otimes \hat{L}^{\otimes n}$, for an ample line bundle L on X. By fully faithfulness $u = \hat{v}$ for a unique morphism $v : \mathcal{O}_X(-m_1)^{r_1} \to \mathcal{O}_X(-m_0)^{r_0}$. Then we get $E =$the completion of the cokernel of v.

b) f quasi projective. We use the extension by zero to reduce to the previous case.

c) General case. Use Chow's lemma and noetherian induction on X □

4.3. **Corollaries.** With the same assumptions of the Existence Theorem we have

(1) $Z \to \widehat{Z}$ gives a bijection between the set of closed subschemes of X which are proper over Y and the set of closed formal subschemes of $\widehat{X}$ which are proper over $\widehat{Y}$.
(2) $Z \to \widehat{Z}$ gives an equivalence between the category of finite schemes over X which are proper over Y and the category formal schemes finite over $\widehat{X}$ which are proper over $\widehat{Y}$.
(3) Assume further that X is proper over Y and Z is noetherian scheme algebraic over Y. Then the application

$$\mathrm{Hom}_Y(Z, Z) \to \mathrm{Hom}_{\widehat{Y}}(\widehat{X}, \widehat{Z}) \ , \quad f \mapsto \hat{f}$$

is bijective.
(4) If $\mathcal{X}$ is a proper, adic $\widehat{Y}$ formal scheme[9]. If there exists an invertible $\mathcal{O}_{\mathcal{X}}$-module L such that L/IL is ample on X_0, then $\mathcal{X}$ is algebrizable.

[8] ...
[9] i.e. $\mathcal{X} = \mathrm{colim}\, \mathcal{X} \times_{\widehat{Y}} Y_n$.

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