# THE WEIGHT-MONODROMY CONJECTURE AFTER PETER SCHOLZE

(NOTES FOR THE ALMOSTXMASSEMINAR)

## 1 The conjecture

We fix a local field $k$ with finite residue field $\mathbb{F}_q$, $q = p^n$ for some prime $p > 0$ (*e.g.* $k = \mathbb{Q}_p, \mathbb{F}_q((t))$). There is a canonical exact sequence of Galois groups

$$1 \to I \to \mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to 1$$

The group $I$ is called the *inertia subgroup* and it has $\mathbb{Z}_l(1)$ as the maximal pro-$l$-quotient, for $l \neq p$ prime. [1]

**Theorem 1.1** (Grothendieck). *Let $\rho : \mathrm{Gal}(\bar{k}/k) \to \mathrm{Gl}(V)$ be a finite dimensional $\bar{\mathbb{Q}}_l$ representation. Then there exists an open subgroup $I_1 \subset I$ such that $\rho$ restricted to $I_1$ is unipotent.*[2]

Let $V(1) := V \otimes_{\mathbb{Z}} \mathbb{Z}_l(1)$ and $t_l : I \to \mathbb{Z}_l(1)$ the projection. A formal consequence of the above theorem is the existence of a unique nilpotent operator $N : V(1) \to V$ such that $\rho(g) = \exp(N \cdot t_l(g))$, for all $g \in I_1$. The map $N$ is called the *logarithm of the nilpotent part of the local monodromy.*

By linear algebra, given a nilpotent endomorphism $N$ on a vector space $V$, then there is a unique (increasing, separated exhaustive) filtration $V_i \subset V_{i+1} \subset ...V$ such that $N^i$ induces an isomorphism $\mathrm{gr}^N_{-i} V \to \mathrm{gr}^N_i V$, where $\mathrm{gr}^N_i := V_i/V_{i-1}$. In our setting this is called the *local monodromy filtration.*

We are interested in the representations arising from geometry, *i.e.* the étale cohomology groups $H^n_{\text{ét}}(X_{\bar{k}}, \bar{\mathbb{Q}}_l)$ associated to an algebraic $k$-scheme $X$. Deligne gave the following

**Conjecture 1.2** (Weight-monodromy (WMC for short), [?]). [3] *Assume $X$ to be proper and smooth over $k$ and $V := H^n_{\text{ét}}(X_{\bar{k}}, \bar{\mathbb{Q}}_l)$. For all $i$ and for any geometric Frobenius element $\phi \in \mathrm{Gal}(\bar{k}/k)$, the eigenvalues of $\phi$ on $\mathrm{gr}^N_i V$ are Weil numbers of weight $n + i$, i.e. algebraic number $\alpha$ s.t. $|\alpha|_\sigma = q^{(i+n)/2}$ for any embedding $\sigma : \bar{\mathbb{Q}} \to \mathbb{C}$.*

**Theorem 1.3** (Deligne, [?]). *The WMC holds true for $X := Y \times_{C\setminus\{x\}} \mathrm{Spec}\, k$, where*

*(1) $C$ be a smooth curve over $\mathbb{F}_q$;*
*(2) $k$ is the (equal characteristic) local field of $C$ at $x \in C(\mathbb{F}_q)$, i.e. $k$ is the quotient field of the henselian local ring $\mathcal{O}^h_{C,x}$ (e.g. $k = \mathbb{F}_q((t))$);*
*(3) $Y$ be a proper and smooth scheme over $C\setminus\{x\}$.*

If char$k =$ the WMC is proved only in a few case before [?].

**Theorem 1.4** (Scholze). *Let $X$ be a smooth and proper of a local field $k$ of characteristic $0$. Assume that $X$ is a set theoretic complete intersection in a projective toric variety $T$ over $k$ (e.g. $T = \mathbb{P}^N_k$). Then the WMC holds true for $X$.*

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[1]Here $\mathbb{Z}_l(1)$ denotes the inverse limit $\lim_n \mu_{l^n}(\bar{\mathbb{F}}_q)$, where $\mu_{l^n}(\bar{\mathbb{F}}_q) = \{x \in \bar{F}_q \ : \ x^{l^n} = 1\}$.

[2]*i.e.* for any $g \in I_1$, $(\rho(g) - \mathrm{id})^r = 0$ for some positive integer $r$ (depending of $g$).

[3]There are several formulations. We give the original one. See Ito [?] for a detailed account.

We will give some details of the proof in the remainder of this note.

## 2 Sketch of the proof

The strategy of the proof is to base change to a perfectoid field, so that we can tilt to characteristic $p$ and use the result of Deligne. So first we base change $X$ and $P$ to the perfectoid field

$$K := \text{completion of } k(\varpi^{1/p^\infty})\ , \quad \varpi \in k \quad \text{a uniformizer}$$

so that its tilt $K^\flat$ is the completed perfection of $\mathbb{F}_q((t))$. This procedure is compatible with monodromy and weights.

The idea is to construct an algebraic variety $Z$ over $K^\flat$ such that $H^n(Y_K) \subset H^n(Z)$. To simplify we take $P = \mathbb{P}^N_k$ to the projective space.

Let's assume (for a while) the following adic results:

**A** The adic space $\lim_\phi(\mathbb{P}^N_K)^{\mathrm{ad}}$ is perfectoid with tilt $\lim_\phi(\mathbb{P}^N_{K^\flat})^{\mathrm{ad}}$, where $\phi(x_0 : ... : x_N) = (x_0^p : ... : x_N^p)$. By almost purity (see next section) there is an equivalence of topoi $(\mathbb{P}^N_{K^\flat})^{\mathrm{ad},\sim}_{\text{ét}} \cong (\lim_\phi(\mathbb{P}^N_K)^{\mathrm{ad}})^\sim_{\text{ét}}$. From this we get a projection map of topological spaces (and also étale topoi) $\pi : (\mathbb{P}^N_{K^\flat})^{\mathrm{ad}} \to (\mathbb{P}^N_K)^{\mathrm{ad}}$, which is given on coordinates $\pi(x_0 : ... : x_N) = (x_0^\sharp : ... : x_N^\sharp)$.

**B** (Approximation) Note that the preimage $\pi^{-1}(X_K) \subset (\mathbb{P}^N_{K^\flat})^{\mathrm{ad}}$ is not algebraic[4]. Anyhow for any neighborhood $\widetilde{X} \supset X^{\mathrm{ad}}_K$ there exists an algebraic variety $Z \subset \mathbb{P}^N_{K^\flat}$ such $\dim Z = \dim X$ and $Z^{\mathrm{ad}} \subset \pi^{-1}(\widetilde{X})$.
Moreover $H^n(\widetilde{X}) = H^n(X^{\mathrm{ad}}_K)$.

**C** (adic vs algebraic: étale comparison) For any algebraic variety $Y$ there is a canonical isomorphism of étale cohomology groups $H^n(Y^{\mathrm{ad}}) = H^n(Y)$.

Then we obtain the following commutative digram

$$\begin{tikzcd}
H^n_{\text{ét}}(\mathbb{P}^N_{\bar{K}}, \bar{\mathbb{Q}}_l) \arrow[r] \arrow[d, "\text{iso by } A+C"] & H^n_{\text{ét}}(\widetilde{X}_{\bar{K}}, \bar{\mathbb{Q}}_l) \arrow[r, "\text{iso by } B+C"] \arrow[d] & H^n_{\text{ét}}(X_{\bar{K}}, \bar{\mathbb{Q}}_l) \\
H^n_{\text{ét}}(\mathbb{P}^N_{\bar{K}^\flat}, \bar{\mathbb{Q}}_l) \arrow[r] & H^n_{\text{ét}}(\pi^{-1}(\widetilde{X})_{\bar{K}^\flat}, \bar{\mathbb{Q}}_l) \arrow[r] & H^n_{\text{ét}}(Z_{\bar{K}^\flat}, \bar{\mathbb{Q}}_l)
\end{tikzcd}$$

and we get a map $\alpha^{(n)} : H^n(X_K) \to H^n(Z)$. Up to alteration, we can assume $Z$ to be smooth over $K^\flat$ so that the result of Deligne applies to $H^n(Z)$. Then we can easily conclude in the following way. First one proves that $\alpha^{(2d)}$ is an isomorphism for $d = \dim Z = \dim X$. Then, by Poincaré duality, $\alpha^{(n)}$ is injective for any $n$ (and compatible w.r.t. the monodromy filtration). This completes the proof of the theorem modulo the knowledge of A, B and C.

To get B and C one uses the theory of adic spaces. The approximation result of B is due to Scholze and relies on the combinatorial nature of toric varieties. Namely the fact that the coherent cohomology of a toric variety with coefficients a line bundle can be easily computed (as for the projective space).

In the next section we will give an account on the almost purity theorem, which is essential for point A.

[4] *e.g.* Let $X = V(x_0 + x_1 + x_2) \subset \mathbb{P}^2_k$. Then $\pi^{-1}(X_K)$ is given by the inverse limit (over $n \geq 0$) of

$$V(x_0^{p^n} + x_1^{p^n} + x_2^{p^n}) \subset (\mathbb{P}^2_K)^{\mathrm{ad}}$$

## 3 Almost purity

**Definition 3.1.** Let $K$ be a perfectoid field.

(1) A morphism $f : (R, R^+) \to (S, S^+)$ of affinoid perfectoid $K$-algebras is called *finite étale* if $S$ is a finite étale $R$-algebra with the induced topology and $S^+$ is the integral closure of $R^+$ in $S$. We call $f$ *strongly finite étale* if further $S^{\circ a}$ is a finite étale $R^{\circ a}$-algebra.

(2) A morphism of perfectoid spaces $f : X \to Y$ is called
   (a) *(strongly) finite étale* if there is an affinoid perfectoid cover $V_i$ of $Y$ such that $U_i = f^{-1}V_i$ is affinoid and $(\mathcal{O}_Y(V_i), \mathcal{O}_Y^+(V_i)) \to (\mathcal{O}_X(U_i), \mathcal{O}_X^+(U_i))$ is (strongly) finite étale.
   (b) *étale* if for any $x \in X$ there is an open neighborhood $U$ of $x$ and a factoriazation

$$\begin{tikzcd} X \arrow[d, "f"'] & U \arrow[l] \arrow[r, "\iota"] \arrow[d, "f_{|U}"'] & W \arrow[dl, "\pi"] \\ Y & V \arrow[l] & \end{tikzcd}$$

   where $\iota$ is open and $\pi$ is (strongly) finite étale. [5]

*Remark* 3.2. Let $f : X \to Y$ be a morphism of perfectoid spaces. By tilting[6] we have that

(1) $f$ is strongly (finite) étale $\iff$ its tilt $f^\flat : X^\flat \to Y^\flat$ is strongly (finite) étale. Note that in characteristic $p$ strongly (finite) étale = (finite) étale.

(2) if $f$ is a strongly finite étale morphism of perfectoid spaces, then for any open affinoid perfectoid $V \subset Y$, its preimage $U$ is affinoid perfectoid, and

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is strongly finite étale. This second statement is much more delicate and uses, apart from tilting, a reduction to the $p$-finite case by a limit argument.

Now we can state the Scholze's generalization of the almost purity theorem.

**Theorem 3.3.** *Let $X$ be a perfectoid affinoid space. Then for any $U \subset X$ open perfectoid affinoid subspace there is an equivalence of categories*

$$\{f : V \to U \text{ strongly finite étale}\} \cong \{\text{finite étale } \mathcal{O}_X(U)\text{-algebras}\}$$

*given by taking global sections: $(f : V \to U) \mapsto \mathcal{O}_V(V)$.*

*Sketch of proof.* The hard part to prove is the essential surjectivity of the functor. Namely let $U = X = \mathrm{Spa}(R, R^+)$ perfectoid affinoid and take a finite étale $R$-algebra $S$. We have to prove that $S^{\circ a}$ is a finite étale $R^{\circ a}$-algebra. The strategy is to reduce to the case of fields where the result is true.

[5] Actually one defines étale maps in the usual way so to be compatible with the definition in Tate setting, [, Proposition 1.7.11]. Anyhow by [, Lemma 2.2.8] we get the equivalence with the definition given here.

[6] **tilt** verb. (1) move or cause to move into a sloping position : [ intrans. ] the floor tilted slightly figurative the balance of industrial power tilted toward the workers ; [ trans. ] he tilted his head to one side.
(2) (figurative) incline or cause to incline toward a particular opinion ; [ intrans. ] he is tilting toward a new economic course.
(3) [ trans. ] move (a camera) in a vertical plane.

First Step: localize. We claim that we can cover $X$ with finitely many rational subsets $U_i$ and that there are strongly étale maps $V_i \to U_i$ such that $\mathcal{O}_{V_i}(V_i) = S \otimes_R \mathcal{O}_X(U_i)$.

So let $x \in X$ be a point. By [**?**, Theorem 3.7] there is an equivalence of categories

$$(a) \qquad \left\{\text{finite étale } /\widehat{k(x)}\right\} \cong \left\{\text{finite étale } /\widehat{k(x^\flat)}\right\}$$

where $k(x^\flat)$ is the residue field of the tilt $X^\flat$ at $x^\flat$ corresponding to $x$.

Recall that $\widehat{k(x)} = \operatorname{colim}_{x\in U} \mathcal{O}_X(U)$, by [**?**, Proposition 2.25]. This implies the following equivalence of categories

$$(b) \qquad \left\{\text{finite étale } /\widehat{k(x)}\right\} \cong 2\text{-}\operatorname*{colim}_{x\in U} \{\text{finite étale } /\mathcal{O}_X(U)\}$$

Combining (a) and (b) we get the following equivalence

$$(a+b) \qquad 2\text{-}\operatorname*{colim}_{x\in U} \{\text{finite étale } /\mathcal{O}_X(U)\} \cong 2\text{-}\operatorname*{colim}_{x^\flat\in U^\flat} \left\{\text{finite étale } /\mathcal{O}_{X^\flat}(U^\flat)\right\}$$

This allows to prove the claim thanks to the properties of strongly étale maps described in the previous remark. $\square$

Second Step: glue. One can prove that the $V_i$ above glue to a perfectoid space $Y$ strongly finite étale over $X$. Again use tilting one can prove that $Y$ is affinoid perfectoid, say $Y = \operatorname{Spa}(T, T^+)$. It remains to show that $T = S$. This follows from the sheaf property of $\mathcal{O}_Y$ and $\mathcal{O}_X$, [**?**, Proposition 6.14], and the flatness of $S$ over $R$.

**Definition 3.4.** Let $X$ be a perfectoid space. The *étale site* of $X$ is the category $X_{\text{ét}}$ of perfectoid spaces which are étale over $X$, and coverings the topological ones.

**Corollary 3.5.** *Let $X$ be a perfectoid space over $K$ with tilt $X^\flat$. Then the tilting operation induces a functorial isomorphism of sites $X_{\text{ét}} \cong X^\flat_{\text{ét}}$.*